Contents

1 Introduction

We explore the long-term distribution properties of progressively dilating shapes in a folded ambient space. Let S be a compact connected Riemannian manifold and N its Riemannian covering space with projection $\pi : N \to S$. Here, S represents the folded version of N. A homothety on N is a diffeomorphism that scales the Riemannian metric by a scalar, with isometries corresponding to a scaling factor of 1. Let $(h_t)_{t>0}$ be a family of homotheties on N whose scaling factors diverge as t increases, and let m_N be a Borel probability measure on N, representing the shape we wish to study under dilation. The push-forward of m_N by h_t, denoted $m_{N,t}$, induces a measure m_t on S.
The central question is to determine the limit points of $(m_t)_{t>0}$ under weak* convergence of probability measures on S as $t \to \infty$. Specifically, we seek sufficient geometric conditions under which m_t equidistributes on S, converging to the renormalized volume measure Vol_S.
The classical questions as formulated above is studied in zero-curvature setting of Euclidean space and tori by the Fourier analysis. And In the hyperbolic setting, where S is a compact hyperbolic d-manifold and $N = \mathbb{H}^d$, the universal covering space, Randol's work [4] provides insight into this problem. Here, homotheties are defined by fixing a base point in $\mathbb{H}^d$ and scaling geodesics accordingly. For more details about the history development on dilating orbits problems, we kindly point the reader to Corso and Davotti [1].
In this paper, we generalize the problem to real rank-1 connected semisimple Lie groups with finite center. Specifically, we focus on the group $SO(n, 1)$. For more details on the structure of this group, refer to Section 2. Let $\Gamma \subset SO(n, 1)$ be a cofinite lattice, meaning that Γ is a discrete subgroup such that the quotient space $\Gamma\backslash SO(n, 1)$ has finite volume.

1.1 The setup: circles in n-dimensional hyperbolic space and their unit tangent bundles

Let Γ be a cocompact lattice of G, where G is the connected component of Lie group $SO(n, 1)$, such that the quotient space $\Gamma\backslash G$ is compact. Later we indicate this homogeneous space with M. Let $L^2(M)$ be the Hilbert space of complex valued functions on M whose modulus is square-integrable with respect to the measure vol, and denote by

$$\langle \phi, \psi \rangle = \int_M \phi\overline{\psi}\, d\mathrm{vol},$$

which gives the inner product in the Hilbert space $L^2(M)$.

And we recall the Casimir element Ω as a second order differential operator acting on $C^\infty(M)$, which is the space of smooth functions over M. The expression of the differential operator $\mathcal{L}_\Omega$ defined by Ω is discussed later in the section 3.1. And it is well-known that the spectrum of Ω is discrete since the homogeneous space M is compact. The spectrum of Casimir operator and the classification of unitary irreducible representation is discussed in details in the section 3.2. For a unitary spherical representation (π, H, G), we denote by Spec_p the spectrum of the representation π corresponding to the principal series and by Spec_c the spectrum of the representation π corresponding to the complementary series. For a cocompact lattice Γ, such that $H = L^2(\Gamma\backslash G)$, the representation π only has a discrete spectrum. We denote by Σ_p the discrete subset of Spec_p and by Σ_f the finite subset of the complementary series and the trivial representation.
We are interested in the quantitative equidistribution properties of expanding circles over the homogeneous space M. As mentioned above, the representation (π, H, G) has a discrete spectrum; therefore, it can be decomposed unitarily as

$$\pi = \bigoplus_{\mu\in\Sigma_p\cup\Sigma_f} \pi_\mu, \quad H = \bigoplus_{\mu\in\Sigma_p\cup\Sigma_f} H_\mu.$$

Then, for any vector $v \in H$, there exists an orthogonal decomposition into a sum of Casimir eigenvectors, expressed as

$$v = \sum_\mu v_\mu.$$

We denote by $C^k(G)$ the space of continuous functions $f : G \to H$, where f is of class C^k. As same as the spectral decomposition of v, the function f can be decomposed into a sum of eigenfunctions f_μ:

$$f = \sum_\mu f_\mu.$$

Before we establish the result for a general function f, we start by considering an irreducible unitary representation (π, H, G). Our goal is to study the spherical average $I(a_t, f)$, defined by

$$I(a_t, f) = \int_K f(a_t k)\, dk.$$

We note that $\int_K f(k)\, dk$ is a K-invariant vector. When H has no non-trivial K-invariant vectors, the average satisfies $I(a_t, f) = 0$. Thus, we are specifically interested in the irreducible unitary spherical representations of G.
By using the decomposition of the unitary representation (π, H, G), we obtain the general result for the average $I(a_t, f)$. In the following subsection, we will provide a precise asymptotic expansion of $I(a_t, f)$ as $t \to \infty$, first an imprecise formula for the eigenfunction

of the Casimir operator $\mathcal{L}_\Omega$ and then the main theorem for general functions satisfying certain regularity conditions.

1.2 Quantitiative equidistribution of expanding circles

This section is dedicated to presenting the main theorem of the draft. We begin by defining the Sobolev norm, and the properties of the Sobolev norm are presented in section 3.2.3. Consider a representation (π, H, G) and $k \in \mathbb{Z}_{\geq 0}$. We denote by $C^k(H)$ the space of vectors $v \in H$ that is of class C^k. Let $C^\infty(H)$ be the space of smooth vectors, which is dense in the Hilbert space H.
For $\phi \in C^\infty(H)$, we define the Sobolev norm with respect to the basis of the Lie algebra $\mathfrak{g}$ of G. We also introduce the second-order linear operator

$$\mathcal{L}_\Delta = \mathcal{L}_{\Omega - 2\Omega_K},$$

which is elliptic and self-adjoint. For more details about its ellipticity and self-adjointness, we refer to Theorem 8.7 in the book [3].

Definition 1. For any $s \in \mathbb{R}_{\geq 0}$, the L^2-Sobolev space $W_s(H)$ of order s is defined to be the domain of the operator $(1 + \mathcal{L}_\Delta)^{s/2}$. The space $W_s(H)$ is equipped with an inner product given by

$$\langle \phi, \psi \rangle_{W_s} = \langle (1 + \mathcal{L}_\Delta)^s \phi, \psi \rangle, \quad \text{for all } \phi, \psi \in C^\infty(H),$$

where 1 denotes the identity operator on H.

Definition 2. Define the Sobolev space $W_s(G)$ for a H-valued function $f : G \to H$ by requiring that$(1 + \mathcal{L}_\Delta)^{s/2} f$ is square-integrable with respect to the Haar measure on G, with the inner product

$$\langle f, g \rangle_{W_s(G)} = \left\langle (1 + \mathcal{L}_\Delta)^{s/2} f, (1 + \mathcal{L}_\Delta)^{s/2} g \right\rangle_{L^2(G,H)}.$$

This is analogous to the vector case; here the operator $(1+\mathcal{L}_\Delta)^{s/2}$ acts on the G-variable. The definition carries over provided that the Laplacian (or Casimir operator) $\mathcal{L}_\Delta$ is defined on H-valued functions on G and has the appropriate domain, which is well-defined because the Casimir operator is in the centre of the universal enveloping algebra.
We begin with the case of eigenfunctions f of the Casimir operator $\mathcal{L}_\Omega$ and denote the Casimir eigenvalue as μ, i.e. $\mathcal{L}_\Omega f = \mu f$. Then the imprecise asymptotic formula of the average $I(a_t, f)$ that we obtain is

$$I(a_t, f) = e^{\lambda_- t} \Theta^- + e^{\lambda_+} \Theta^+ + \mathcal{R}(f, t)$$

where the constants $\Theta^{\pm}(f)$ is dominated by the Sobolev norm of f with suitable order and the error term $\mathcal{R}(f,t) = o(e^{\lambda_{\pm}t})$. And the precise formula of the constants $\Theta^{\pm}(f)$ and the error term $\mathcal{R}(f,t)$ is in the theorem 5.1, which depends the classification of the unitary irreducible representations.

For the statement of the main theorem, we define the index functionh δ_i with $i = 0, 1, 2$ is defined as follows:

$$\delta_0 = \begin{cases} 1, & \text{if } \mu = -\frac{(n-1)^2}{4} \in \Sigma_p, \\ 0, & \text{if } \mu = -\frac{(n-1)^2}{4} \notin \Sigma_p. \end{cases}, \delta_1 = \begin{cases} 1, & \text{if } \mu = 1 - \frac{(n-1)^2}{4} \in \Sigma_f, \\ 0, & \text{if } \mu = 1 - \frac{(n-1)^2}{4} \notin \Sigma_f. \end{cases}, \delta_2 = \begin{cases} 1, & \text{if } \mu = 2 - \frac{(n-1)^2}{4} \in \Sigma_f, \\ 0, & \text{if } \mu = 2 - \frac{(n-1)^2}{4} \notin \Sigma_f. \end{cases}$$

Theorem 1.1. *Given an integer $l = \lfloor \frac{\nu}{2} \rfloor + 1$ and a unitary spherical representation (π, H) of the Lie group G, for any function $f \in W_{2k+2l+3}(G)$, there exist vectors satisfying*

$$\left\|\Theta^{\pm}(f)\right\| \ll_{\nu} \|f\|_{W_{2k+3}}, \quad \left\|\Theta^{(i)}_{\pm,-2i,t^{\mathfrak{i}}}\right\| \ll_{\nu} \|f\|_{W_{2k+3+2i}}$$

for $i = 0, \ldots, l$ and $\mathfrak{i} = 0, 1, 2$. When $\mathfrak{i} = 0$, it is omitted from the notation.
For $t \geq 1$, the spherical average $I(a_t, f)$ satisfies the asymptotic formula

$$\begin{aligned} \int_K f(a_t k)\, dk &= \sum_{\mu \in \Sigma_p} \sum_{\varpi} \int_K f_{\mu,\varpi}(a_t k)\, dk + \sum_{\mu \in \Sigma_f} \sum_{\varpi} \int_K f_{\mu,\varpi}(a_t k)\, dk \\ &= \int_K f(k)\, dk + I(a_{t,f_p}) + I(a_t, f_c) + \mathcal{R}_{all}(f,t). \end{aligned}$$

where the asymptotic formula for the principal series $I(a_t, f_p)$ is given by

$$\begin{aligned} I(a_t, f_p) = & \sum_{\nu \in i\mathbb{R}_{>0}} \sum_{\varpi} \left(e^{\lambda_- t}\Theta^-(f_{\nu,\varpi}) + e^{\lambda_+ t}\Theta^+(f_{\nu,\varpi})\right) \\ & + \delta_0 \Bigg(e^{\frac{1}{2}(1-n)t}\Theta^-(f) + te^{\frac{1}{2}(1-n)t}\Theta^+(f) \\ & \quad + e^{-\frac{1}{2}(3+n)t}(1+4t)\frac{1}{4}\sum_{s=2}^{n}\Theta^-(N^2_{n_{s-1}}f) \\ & \quad + e^{-\frac{1}{2}(3+n)t}(4t^2+3t+1)\frac{1}{4}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}}f) \Bigg). \end{aligned}$$

The asymptotic formula for the complementary series $I(a_t, f_c)$ is given by

$$
\begin{aligned}
I(a_t, f_c) = &\sum_{\nu\in(0,1)} \sum_{\varpi} \Bigg(e^{\lambda_- t}\Theta^-(f_{\nu,\varpi}) + e^{\lambda_+ t}\Theta^+(f_{\nu,\varpi}) \\
&+ e^{(\lambda_- -2)t}\sum_{s=2}^{n}\Theta^-(N^2_{n_{s-1}} f_{\nu,\varpi})P^-(\nu) + e^{(\lambda_+ -2)t}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}} f_{\nu,\varpi})P^+(\nu)\Bigg) \\
&+ \sum_{\nu\in(1,2)} \sum_{\varpi} \left(e^{\lambda_+ t}\Theta^+(f) - e^{(\lambda_+ -2)t}\frac{1}{2\nu-2}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}} f)\right) \\
&+ \sum_{\nu\in\left(2,\frac{n-1}{2}\right)} \sum_{\varpi}\sum_{i=0}^{l} e^{(\lambda_+ -2i)t}\Theta^{(l)}_{+,-2i}(f) \\
&+ \delta_1\left(e^{\lambda_- t}\Theta^{(2)}_-(f) + e^{\lambda_+ t}\Theta^{(2)}_+(f) - e^{\lambda_- t}(t-1)\Theta^{(2)}_{-,t}\right) \\
&+ \delta_2\left(\sum_{i=0}^{2} e^{(\lambda_+ -2i)t}\Theta^{(2)}_{+,-2i}(f) + e^{\lambda_- t}\Theta^{(2)}_-(f) + e^{\lambda_- t}(t-1)\Theta^{(2)}_{-,t}\right).
\end{aligned}
$$

Finally, the error term $\mathcal{R}_{all}(f,t)$ satisfies the bound

$$\|\mathcal{R}_{all}(f,t)\| \ll_\nu e^{-\frac{1}{2}(3+n)t}\,\|f\|_{W_{2k+2l+3}},$$

where $\ll_\nu$ depends the spectral parameter ν.

2 Expression of the Casimir operator $\mathcal{L}_\Omega$ for semisimple connected Lie group

This section is devoted to determining the exact expression of the Casimir operator $\mathcal{L}_\Omega$ for a semisimple connected Lie group. This calculation is more general than that for the specific case of the Lie group $SO(n,1)$.

Given a connected semisimple Lie group with finite center with a maximal compact subgroup $K \subset G$, the Cartan subgroup compatible with K is denoted as A. We consider the following spherical averages: let $(\pi, \mathcal{H})$ be a unitary representation of G. For a vector $v \in \mathcal{H}$ and $a \in A$, the spherical average is defined as

$$I(a) = \int_K \pi(a_t k)v\,dk.$$

The goal is to obtain an asymptotic formula for the average $I(a)$. Eventually, we specialize it to $\mathcal{H} = L^2(\Gamma\backslash G)$ and get pointwise bounds in terms of suitable Sobolev norms.

The main idea: for any irreducible representation, the average $I(a)$ satisfies a system of linear differential equations.

We denote $\mathfrak{g}$ by the Lie algebra of G. Fix a maximal compact subgroup $K \subset G$ and the corresponding Cartan involution θ of Lie algebra $\mathfrak{g}$, we have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \otimes \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of K. Let $B(x, y) = \mathrm{Tr}(\mathrm{ad}_X \cdot \mathrm{ad}_Y)$ be the Killing form on $\mathfrak{g}$. The Killing form B is negative definite on $\mathfrak{k}$ and positive-definite on $\mathfrak{p}$. Furthermore, $\mathfrak{k}$ and $\mathfrak{p}$ are orthogonal. Then

$$< X, Y >= -B(X, \theta(Y))$$

defines a positive-definite inner product on $\mathfrak{g}$. Fix an orthogonal basis $\{K_i\}$ of $\mathfrak{k}$ and an orthogonal basis $\{P_j\}$. Then $\{K_i\} \cup \{P_j\}$ is an orthogonal basis of $\mathfrak{g}$, and $\{-K_i\} \cup \{P_j\}$ is the dual basis.
Then the Casimir element is defined by:

$$\Omega = \sum_i -K_i^2 + \sum_j P_j^2 \in U(\mathfrak{g})$$

It is known that the definition of Ω does not depend on a choice of the orthogonal basis of $\mathfrak{g}$.

2.1 The Casimir operator

The goal of this section is to find a convenient formula for the Casimir operator Ω.
Define the right translation as

$$R_g f(h) = f(hg), \;\; h, g \in G$$
$$R_X f(h) = \frac{d}{dt}[f(h \exp(tX))] \mid_{t=0}, \;\; X \in \mathfrak{g}.$$

It is known that $X \to R_X$ is a Lie algebra homeomorphism of the universal enveloping algebra $U(\mathfrak{g})$. So it extends to a homomorphism of the universal enveloping algebra. Similarly, one defines left-invariant operators L_g for $g \in G$ and corresponding L_X for $X \in U(\mathfrak{g})$. Since the Casimir operator Ω is in the center of $U(\mathfrak{g})$, $R_\Omega = L_\Omega$. Given a semisimple Lie algebra $\mathfrak{g}$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, and a fixed maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$, then $\mathrm{ad} \mid_{\mathfrak{a}}$ is self-adjoint and diagonalizable. We define a root as a non-zero linear functional $\alpha \in \mathfrak{a}^*$ (the dual space of $\mathfrak{a}$). The set $\mathfrak{g}_\alpha$ is called the root space corresponding to the root α, and it is defined as follows:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_Y(X) = [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{a}\}.$$

This set $\mathfrak{g}_\alpha$ consists of all elements X in $\mathfrak{g}$ that are eigenvectors for the adjoint action of every element $Y \in \mathfrak{a}$, with the corresponding eigenvalue given by $\alpha(Y)$.
Then, $\mathfrak{g}$ is decomposed into a direct sum of root spaces

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

Note that different root spaces are orthogonal with respect to Killing form.
Fix an order on $\mathfrak{a}^*$ and denote by Σ^+ by the corresponding set of positive roots. Let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

Then the Lie algebra $\mathfrak{g}$ can be decomposed as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Also, the Lie group G can be decomposed as KAN where $A = \exp(\mathfrak{a})$ and $N = \exp(n)$. This is known as the Iwasawa decomposition.
Moreover, the product map is known to be a diffeomorphism. We have $\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Fix root vectors n_i with norm $||n_i|| = 1$, with positive roots α_i, so that they form an orthonormal basis of $\mathfrak{n}$. Note that $\theta(n_i)$ are also orthonormal vectors.
Further, let $\{m_j\}$ be an orthonormal basis of $\mathfrak{m}$ and $\{a_l\}$ be an orthonormal basis of $\mathfrak{a}$. Then we have a complete orthonormal basis of $\mathfrak{g}$:

$$\{n_i\} \cup \{\theta(n_i)\} \cup \{m_j\} \cup \{a_l\}.$$

Let

$$k_i = \frac{n_i + \theta(n_i)}{\sqrt{2}} \in \mathfrak{k}$$
$$p_i = \frac{n_i - \theta(n_i)}{\sqrt{2}} \in \mathfrak{p}.$$

Then $\{k_i\} \cup \{p_i\} \cup \{m_j\} \cup \{a_l\}$ is an orthonormal basis of $\mathfrak{g}$.
Now the Casimir operator can be written as

$$\begin{aligned}\Omega &= \sum_i -k_i^2 + \sum_j -m_j^2 + \sum_l a_l^2 + \sum_i p_i^2 \\ &= \Omega_K + \sum_l a_l^2 + \sum_i p_i^2,\end{aligned}$$

where Ω_K is the Casimir operator of K.
From now on, we denote

$$x_i = \begin{cases} a_i, & 1 \le i \le r = \dim(\mathfrak{a}) \\ p_{i-r}, & r < i \le \dim(\mathfrak{p}) = d. \end{cases}.$$

Then $\{x_i\}$ is an orthonormal basis of $\mathfrak{p}$.

2.2 Explicit formula of $\mathcal{L}_\Omega$

Now the goal is to find a formula for $\mathcal{L}_\Omega$ in terms of Iwasawa decomposition. Here, with a bit of abuse of the notations, we use the letter $x \in \mathfrak{g}$ for the element of Lie algebra and the corresponding vector field.

For $g \in G$, we write $g = k_g a_g n_g$ for its Iwasawa decomposition. The subscript is omitted if it is clear in the context. For a smooth function $f \in C^\infty(G)$ and $g = kan \in G$. We introduce the notation of differential operators as follows:

$$\begin{aligned}
\text{For } x \in \mathfrak{k}, K_x f(g) &= \frac{d}{dt}[f(\exp(tx)kan)]\Big|_{t=0} \\
\text{For } x \in \mathfrak{a}, A_x f(g) &= \frac{d}{dt}[f(k\exp(tx)an)]\Big|_{t=0} \\
\text{For } x \in \mathfrak{n}, N_x f(g) &= \frac{d}{dt}[f(ka\exp(tx)n)]\Big|_{t=0}
\end{aligned}$$

Note that those differential operators commute.

Lemma 1. *For $g = kan \in G$, define $\omega_{ij} =< \mathrm{Ad}(k)X_i, X_j >$. Since $\mathrm{Ad}(k)$ is an orthogonal transformation, (w_{ij}) is an orthogonal matrix. Also, $\omega_{ij}(k^{-1}) = \omega_{ji}(k)$.*

First, we want to compute $\mathcal{L}_{x_i}$. Let $g = kan \in G$, then

$$\begin{aligned}
\mathcal{L}_{x_i} f(g) &= \frac{d}{dt}[f(\exp(tx)kan)]\Big|_{t=0} \\
&= \frac{d}{dt}[f(\exp(t\mathrm{Ad}(k^{-1})x)an)]\Big|_{t=0}.
\end{aligned}$$

We have explicit expansion of $\mathrm{Ad}(k^{-1})x_i$,

$$\begin{aligned}
\mathrm{Ad}(k^{-1})x_i &= \sum_j \omega_{ij}(k^{-1})x_j \\
&= \sum_j \omega_{ji}(k)x_j
\end{aligned}$$

Plugging into the formula of $\mathcal{L}_{x_i}f(g)$, it becomes

$$
\begin{aligned}
\mathcal{L}_{x_i}f(g) &= R_{\mathrm{Ad}(k^{-1})x_i}R_{an}f(k) \\
&= \sum_j \omega_{ji}(k)R_{x_j}R_{an}f(k) \\
&= \sum_{j=1}^{r}\omega_{ji}(k)A_{a_j}f(g) + \sum_{j=r+1}^{d}\omega_{ji}(k)R_{p_{j-r}}R_{an}f(k).
\end{aligned}
$$

Recall the definition of p_{j-r}:

$$
\begin{aligned}
k_s &= \frac{n_s+\theta(n_s)}{\sqrt{2}} \in \mathfrak{k} \\
p_s &= \frac{n_s-\theta(n_s)}{\sqrt{2}} \in \mathfrak{p}.
\end{aligned}
$$

So the Cartan involution $\theta(n_s)$ is $\frac{k_s-p_s}{\sqrt{2}}$. and

$$
p_s = \frac{\sqrt{2}n_s - k_s + p_s}{2}.
$$

This gives us $p_s = -k_s + \sqrt{2}n_s$.
Hence the terms in the second summation $R_{p_{j-r}}R_{an}f(k)$ becomes

$$
-K_{k_{j-r}}f(g) + \sqrt{2}R_{n_{j-r}}R_{an}f(k),
$$

where $R_{n_{j-r}}R_{an}f(k)$ can be expanded as follows,

$$
\begin{aligned}
R_{n_{j-r}}R_{an}f(k) &= \frac{d}{dt}[f(k\exp(tn_{j-r})an)]\Big|_{t=0} \\
&= \frac{d}{dt}[f(ka\cdot\exp(\mathrm{Ad}(a^{-1})tn_{j-r})n)] \\
&= e^{-\alpha_{j-r}(\log a)}N_{n_{j-r}}f(g).
\end{aligned}
$$

With the Iwasawa decomposition $g = kan$, we conclude that

$$
\begin{aligned}
\mathcal{L}_{x_i}f(g) = &\sum_{j=1}^{r}\omega_{ji}(k)A_{a_j}f(g) \\
&+ \sum_{j=r+1}^{d}\omega_{ji}(k)[-K_{k_{j-r}}f(g) + \sqrt{2}e^{-\alpha(\log a)}N_{n_{j-r}}f(g)]
\end{aligned}
$$

For simplicity, we write

$$
\mathcal{L}_{x_i}f(g) = -\Delta_i^K f(g) + \Delta_i^A f(g) + \sqrt{2}\Delta_i^N f(g),
$$

where

$$\Delta_i^K f(g) = \sum_{j=r+1}^{d} \omega_{ji}(k) K_{k_{j-r}} f(g),$$
$$\Delta_i^A f(g) = \sum_{j=1}^{r} \omega_{ji}(k) A_{a_j} f(g),$$
$$\Delta_i^N f(g) = \sum_{j=r+1}^{d} \omega_{ji}(k) e^{-\alpha(\log a)} N_{n_{j-r}} f(g).$$

We have

$$\mathcal{L}_{x_i^2} = \mathcal{L}_{x_i}^2 = (-\Delta_i^K + \Delta_i^A + \sqrt{2}\Delta_i^N)^2.$$

Therefore, the next step is to compute all the terms of $\mathcal{L}_{x_i}^2$, let us start on $\sum_i \Delta_i^K \Delta_i^A$. For $y \in \mathfrak{k}$ and $j = 1, \ldots, r$, as above by Iwasawa decomposition, $g = kan$.

$$\begin{aligned}
&K_y[\omega_{ji}(k) A_{a_j} f(kan)] \\
&= \frac{d}{dt}[\omega_{ji}(k\exp(ty)) A_{a_j} f(k\exp(ty)an)]\Big|_{t=0}. \\
&= \frac{d}{dt}[\omega_{ji}(k\exp(ty))\Big|_{t=0} \circ A_{a_j} f(kan)] \\
&\quad + \omega_{ji}(k) K_y A_{a_j} f(kan)
\end{aligned}$$

When $j \in \{1, \ldots, r\}$, $x_j = a_j$. The first derivative that we got can be further calculated as

$$\begin{aligned}
\frac{d}{dt}[\omega_{ji}(k\exp(ty))]\Big|_{t=0} &= \frac{d}{dt}[\langle \mathrm{Ad}(k\exp(ty))x_j, x_i\rangle]\Big|_{t=0} \\
&= \frac{d}{dt}[\langle \exp(ty)a_j, \mathrm{Ad}(k^{-1})x_i\rangle]\Big|_{t=0} \\
&= \langle [y, a_j], \mathrm{Ad}(k^{-1})x_i\rangle
\end{aligned}$$

Then by expressing $[y, a_j]$ in terms of the basis $\{x_l\}$:

$$[y, a_j] = \sum_{l=1}^{n} \langle [y, a_j], x_l\rangle x_l,$$

the formulae becomes

$$\begin{aligned}
\frac{d}{dt}[\omega_{ji}(k\exp(ty))]\Big|_{t=0} &= \sum_l \omega_{il}(k^{-1}) \langle [y, a_j], x_l\rangle \\
&= \sum_l \omega_{li}(k) \langle [y, a_j], x_l\rangle.
\end{aligned}$$

Hence we conclude that

$$\begin{aligned}
&K_y[\omega_{ji}(k)A_{a_j}f(kan)] \\
&=\sum_l \omega_{li}(k)\langle [y,a_j],x_l\rangle A_{a_j}f(kan)+\omega_{ji}(k)K_yA_{a_j}f(kan).
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
\Delta_i^K\Delta_i^A f(g) &= \sum_{s=r+1}^{d}\sum_{j=1}^{r}\omega_{si}(k)K_{k_{s-r}}[\omega_{ji}(k)A_{a_j}f(g)] \\
&= \sum_{s=r+1}^{d}\sum_{j=1}^{r}\sum_l \omega_{si}(k)\omega_{li}(k)\langle [k_{s-r},a_j],x_l\rangle A_{a_j}f(kan) \\
&+ \sum_{s=r+1}^{d}\sum_{j=1}^{r}\omega_{si}(k)\omega_{ji}(k)K_yA_{a_j}f(kan).
\end{aligned}$$

Since $\{\omega_{ij}\}$ is orthogonal as defined in 1,

$$\sum_i \omega_{si}(k)\omega_{li}(k)=\delta_{sl}.$$

So the second term in the above formula of $\Delta_i^K\Delta_i^A$ gives 0, and now taking the summation over all i,

$$\begin{aligned}
\sum_i \Delta_i^K\Delta_i^A f(g) &= \sum_{s=r+1}^{d}\sum_{j=1}^{r}\langle [k_{s-r},a_j],x_s\rangle A_{a_j}f(kan) \\
&= \sum_{s=r+1}^{d}\sum_{j=1}^{r}\langle [k_{s-r},a_j],p_{s-r}\rangle A_{a_j}f(kan).
\end{aligned}$$

Using the invariance of Killing form,

$$\begin{aligned}
\langle [k_{s-r},a_j],p_{s-r}\rangle &= -B([k_{s-r},a_j],p_{s-r}) \\
&= B(a_j,[k_{s-r},p_{s-r}]),
\end{aligned}$$

where

$$\begin{aligned}
[k_{s-r},p_{s-r}] &= \left[\frac{n_{s-r}+\theta(n_{s-r})}{\sqrt{2}},\frac{n_{s-r}-\theta(n_{s-r})}{\sqrt{2}}\right] \\
&= [\theta(n_{s-r}),n_{s-r}].
\end{aligned}$$

Therefore, the killing form becomes

$$\begin{aligned}\langle [k_{s-r}, a_j], p_{s-r}\rangle &= B(a_j, [\theta(n_{s-r}), n_{s-r}])\\ &= -B(a_j, [n_{s-r}, \theta(n_{s-r})])\\ &= B([n_{s-r}, a_j], \theta(n_{s-r}))\\ &= -\alpha_{s-r}(a_j)\cdot\langle n_{s-r}, n_{s-r}\rangle\\ &= -\alpha_{s-r}(a_j)\end{aligned}$$

Denote $2\rho = \sum\limits_{s=r+1}^{d} \alpha_{s-r}$, finally,

$$\sum_i \Delta_i^K \Delta_i^A f(g) = -\sum_{j=1}^{r} 2\rho(a_j) A_{a_j} f(g).$$

Next step is to compute $\sum\limits_i \Delta_i^A \Delta_i^K$. Notice that $\omega_{ji}(k)$ commutes with the derivative A_{a_j}. Using the orthogonal relationship in the lemma 1, $\sum\limits_i \omega_{si}(k)\omega_{li}(k) = \delta_{sl}$, such that

$$\sum_i \Delta_i^A \Delta_i^K f(g) = 0.$$

Furthermore, the computation of $\sum\limits_i \Delta_i^K \Delta_i^N$ is similar to . We get

$$\begin{aligned}&\Delta_i^K \Delta_i^N f(g)\\ &= \sum_{s=r+1}^{d}\sum_{j=r+1}^{d}\sum_l \omega_{si}(k)\omega_{li}(k) e^{-\alpha_{j-r}(\log a)}\langle [k_{s-r}, p_{j-r}], x_l\rangle N_{n_{j-r}} f(g)\\ &+ \sum_{s=r+1}^{d}\sum_{j=r+1}^{d} \omega_{si}(k)\omega_{ji}(k) e^{-\alpha_{j-r}(\log a)} K_{k_{s-r}} N_{n_{j-r}} f(g).\end{aligned}$$

So that the summation over i should become

$$\begin{aligned}&\sum_i \Delta_i^K \Delta_i^N f(g)\\ &= \sum_{s=r+1}^{d}\sum_{j=r+1}^{d} e^{-\alpha_{j-r}(\log a)}\langle [k_{s-r}, p_{j-r}], p_{s-r}\rangle N_{n_{j-r}} f(g)\\ &+ \sum_{s=r+1}^{d} e^{-\alpha_{s-r}(\log a)} K_{k_{s-r}} N_{n_{s-r}} f(g),\end{aligned}$$

where the inner product is dealt as same as above.

$$\begin{aligned}\langle [k_{s-r}, p_{j-r}], p_{s-r}\rangle &= B([k_{s-r}, p_{j-r}], p_{s-r})\\ &= -B(p_{j-r}, [k_{s-r}, p_{s-r}]),\end{aligned}$$

And we compute that the Lie bracket $[k_{s-r}, p_{s-r}] = [\theta(n_{s-r}), n_{s-r}]$. Note that this element lies in $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. In particular, this element is orthogonal to p_{j-r}. Hence, the inner product $\langle [k_{s-r}, p_{j-r}], p_{s-r}\rangle = 0$, and

$$\sum_i \Delta_i^K \Delta_i^N f(g) = \sum_{s=r+1}^{d} e^{-\alpha_{s-r}}(\log a) K_{k_{s-r}} N_{n_{s-r}} f(g).$$

Now we start to compute the next term $\sum_i \Delta_i^N \Delta_i^K$.

$$\begin{aligned}&\Delta_i^N \Delta_i^K f(g)\\ =&\sum_{s=r+1}^{d}\sum_{j=r+1}^{d} \omega_{si}(k)\omega_{ji}(k) e^{-\alpha_{j-r}(\log a)} N_{n_{j-r}} K_{k_{s-r}} f(g)\end{aligned}$$

Again, by the orthogonal relationship 1, we obtain the desiring result

$$\sum_{i=1}^{d} \Delta_i^N \Delta_i^K = \sum_{s=r+1}^{d} e^{-\alpha_{s-r}(\log a)} N_{n_{s-r}} K_{k_{s-r}} f(g).$$

Similarly, due to the orthogonal relationship 1,

$$\begin{aligned}&\sum_i^d \Delta_i^N \Delta_i^A\\ =&\sum_{i=1}^{d}\sum_{s=r+1}^{d}\sum_{j=1}^{r} \omega_{si}(k)\omega_{ji}(k) e^{-\alpha_{s-r}(\log a)} N_{n_{s-r}} A_{a_j} f(g)\\ =&0\end{aligned}$$

The next mission is to compute the term $\sum_i^d \Delta_i^A \Delta_i^N$. For any $y \in \mathfrak{a}$, we compute the composition of two differential operators A_y and N_{n_s},

$$\begin{aligned}&A_y[e^{-\alpha_s(\log a)} N_{n_s} f(g)]\\ =&A_y(e^{-\alpha_s \log a}) + e^{-\alpha_s \log a} A_y N_{n_s} f(g).\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta_i^A \Delta_i^N f(g)\\ =&-\sum_{s=r+1}^{d}\sum_{j=1}^{r} \omega_{si}(k)\omega_{ji}(k)\alpha_{s-r}(a_j) e^{-\alpha_{s-r}(a)} A_{a_j} N_{n_{s-r}} f(g)\\ &+\sum_{s=r+1}^{d}\sum_{j=1}^{r} \omega_{si}(k)\omega_{ji}(k) e^{-\alpha_{s-r}(\log a)} A_{a_j} N_{n_{s-r}} f(g).\end{aligned}$$

So $\sum_{i=1}^{d} \Delta_i^A \Delta_i^N f(g)$ vanishes because of orthogonal relationship 1.
At the end, we need to calculate the square terms $(\Delta_i^K)^2$, $(\Delta_i^N)^2$ and $(\Delta_i^A)^2$. By the similar process as above, we obtain

$$\begin{aligned}
&(\Delta_i^K)^2 f(g) = \Delta_i^K \Delta_i^K f(g) \\
&= \sum_{s=r+1}^{d} \sum_{j=r+1}^{d} \omega_{si}(k) K_{k_{s-r}} [\omega_{ji}(k) K_{k_{j-r}} f(g)] \\
&= \sum_{s=r+1}^{d} \sum_{j=r+1}^{d} \sum_{l} \omega_{si}(k) \omega_{li}(k) \langle [k k_{s-r}, k_{j-r}], x_l \rangle K_{k_{j-r}} f(g) \\
&+ \sum_{s=r+1}^{d} \sum_{j=r+1}^{d} \sum_{l} \omega_{si}(k) \omega_{li}(k) K_{k_{s-r}} K_{k_{j-r}} f(g),
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_i (\Delta_i^K)^2 f(g) \\
&= \sum_{s=r+1}^{d} \sum_{j=r+1}^{d} \langle [k_{s-r}, k_{j-r}], p_{s-r} \rangle K_{k_{j-r}} f(g) \\
&+ \sum_{s=r+1}^{d} K_{k_{s-r}}^2 f(g).
\end{aligned}$$

The inner product $\langle [k_{s-r}, k_{j-r}], p_{s-r} \rangle = -B(k_{j-r}, [k_{s-r}, p_{s-r}])$, where the Lie bracket $[k_{s-r}, p_{s-r}] = [\theta(n_{s-r}), n_{s-r}]$. Note that this element lies in $\mathfrak{g}_0 \cap \mathfrak{p} = \mathfrak{a}$. In particular, this element is orthogonal to p_{j-r}. Hence, the inner product $\langle [k_{s-r}, k_{j-r}], p_{s-r} \rangle = 0$, and

$$\begin{aligned}
\sum_i (\Delta_i^K)^2 f(g) &= \sum_{s=r+1}^{d} K_{s-r}^2 f(g) \\
&= \sum_{s=r+1}^{d} R_{an} R_{k_{s-r}}^2 f(k) \\
&= R_{an} R_{-\Omega_K} f(k) + R_{an} R_{\Omega_M} f(k) \\
&= \mathcal{L}_{-\Omega_K} f(g) + K_{\Omega_M} f(g).
\end{aligned}$$

Then we compute $\sum_i (\Delta_i^A)^2$.

$$(\Delta_i^A)^2 f(g) = \sum_{s=1}^{r} \sum_{j=1}^{r} \omega_{si}(k) \omega_{ji}(k) A_{a_s} A_{a_j} f(g).$$

Thanks to the orthogonal relationship 1, we get

$$\sum_{i=1}^{d}(\Delta_i^A)^2 f(g) = \sum_{s=1}^{r}(A_{a_s})^2 f(g).$$

Finally, the last term is $\sum_i (\Delta_i^N)^2$.

$$(\Delta_i^N)^2 f(g) = \sum_{s=r+1}^{d}\sum_{j=r+1}^{d} \omega_{si}(k)\omega_{ji}(k) e^{-\alpha_{s-r}(\log a)-\alpha_{j-r}(\log a)} N_{n_{s-r}} N_{n_{j-r}} f(g),$$

Once again, by the orthogonal relationship 1, the formula simplifies to

$$\sum_{i=1}^{d}(\Delta_i^N)^2 f(g) = \sum_{s=r+1}^{d} e^{-2\alpha_{s-r}(\log a)} N_{n_{s-r}}^2 f(g).$$

Now we are ready to assemble all those terms together. Recall that

$$\Omega = \Omega_K + \sum_{i=1}^{d} X_i^2,$$
$$\mathcal{L}_{X_i}^2 = (-\Delta_i^K + \Delta_i^A + \sqrt{2}\Delta_i^N)^2.$$

Hence we obtain the final theorem,

Theorem 2.1. *For $g = kan \in G$, the derivative formula of the Casimir operator is given as follows*

$$\begin{aligned}\mathcal{L}_\Omega f(g) = &\sum_{i=1}^{r}[A_{a_i}^2 f(g) + 2\rho(a_i) A_{a_i} f(g)] \\ &+ \sum_{s=r+1}^{d}[-2\sqrt{2}e^{-\alpha_{a-r}(\log a)} K_{k_{s-r}} N_{n_{s-r}} f(g) \\ &+ 2e^{-\alpha_{s-r}(\log a)} N_{n_{s-r}}^2 f(g)] + K_{\Omega_M} f(g)\end{aligned}$$

3 Preliminaries on Harmonic analysis on $SO(n,1)$

3.1 Structure of Orthogonal groups

This section revisits fundamental concepts pertaining to the Lie group $SO(n,1)$ and its associated Lie algebra $\mathfrak{so}(n,1)$. The objective is to derive the explicit expression of the Casimir operator $\mathcal{L}_\Omega$ for the group $SO(n,1)$. The Lie group $SO(n,1)$, also known as the special orthogonal group of signature $(n,1)$, is defined as the group of all linear transformations that preserve the quadratic form

$$Q(x) = x_1^2 + x_2^2 + \ldots + x_n^2 - x_{n+1}^2$$

and have determinant equal to 1. Formally, $SO(n,1)$ consists of all $(n+1)$ Square matrices A that satisfy the following conditions:

$$A^T J A = J \quad \text{and} \quad \det(A) = 1,$$

where J is the diagonal matrix given by

$$J = \mathrm{diag}(1, 1, \ldots, 1, -1),$$

A^T denotes the transpose of A, and $\det(A)$ is the determinant of A.
In other words, if x is a column vector in $\mathbb{R}^{n+1}$, then a matrix A belongs to $SO(n,1)$if it satisfies

$$Q(Ax) = Q(x)$$

for all such vectors x. The group $SO(n,1)$ is a non-compact Lie group that acts isometrically on n-dimensional hyperbolic space, which can be thought of as a pseudo-Riemannian manifold with constant negative curvature.
The Lie algebra $\mathfrak{g} = \mathfrak{so}(n,1)$ is defined as

$$\mathfrak{g} = \{X : X^t J + JX = 0 \text{ and } \mathrm{Tr}(X) = 0\},$$

In this draft, we assume $n \geq 4$. For a matrix X in the Lie algebra $\mathfrak{so}(n,1)$, it should be of form

$$\begin{pmatrix} 0 & -v & x \\ v^T & A & w^T \\ x & w & 0 \end{pmatrix} \tag{1}$$

where $A^T = -A$, $v, w \in \mathbb{R}^{n-1}$ and $x \in \mathbb{R}$. And Cartan involution is defined as

$$\theta(X) = -X^T,$$

such that the Cartan decomposition of the Lie algebra with respect to the involution is given as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that

$$\theta(\mathfrak{k}) = \mathfrak{k}, \theta(\mathfrak{p}) = -\mathfrak{p}.$$

where

$$\mathfrak{p} = \left\{ m \in \mathfrak{g} \mid m = \begin{pmatrix} 0 & 0 & h \\ 0 & 0 & w \\ h & w^T & 0 \end{pmatrix} \right\}, \mathfrak{k} = \left\{ \begin{pmatrix} \mathfrak{K} & 0 \\ 0 & 0 \end{pmatrix} \middle| \mathfrak{K} \in \mathfrak{so}(n), \mathfrak{K}^T = -\mathfrak{K} \right\}$$

And we define a Cartan subgroup

$$\mathfrak{a} = \left\{ \mathrm{a}_t = \begin{pmatrix} 0 & \cdots & 0 & t \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ t & \cdots & 0 & 0 \end{pmatrix} \middle| h \in \mathbb{R} \right\},$$

then the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ is

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid C^T = -C, C \in M_{n-1}(\mathbb{R}) \right\}.$$

Since $\mathfrak{a}$ is one-dimensional, there is only one simple root that defines the roots. For an element $H \in \mathfrak{a}$, a root is expressed as a linear functional $\alpha : \mathfrak{a} \to \mathbb{R}$, defined through the adjoint representation. Specifically, a non-zero element $\alpha \in \mathfrak{a}^*$ is considered a root if

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{ad}_Y X = \alpha(Y)X, \text{ for all } Y \in \mathfrak{a}\},$$

where $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ denotes the adjoint representation, defined by $\mathrm{ad}_Y X = [Y, X]$. By fixing the basis

$$\mathrm{a}_1 = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \end{pmatrix},$$

we can, without loss of generality, define the root $\alpha(\mathrm{a}_1)$ as 1, which implies that $\alpha(\mathrm{a}_t) = t$. The set of roots is $\{\alpha, -\alpha\}$. Given the fixed basis a_1, for a Lie algebra element $\mathrm{a}_t \in \mathfrak{a}$, corresponding to the Lie group element $a_t = \exp(\mathrm{a}_t)$, we have

$$a_t = \begin{pmatrix} 0 & & & t \\ & \ddots & & \vdots \\ & & 0 & 0 \\ t & \cdots & 0 & 0 \end{pmatrix}.$$

Next, we compute $\mathrm{ad}_h X = [h, X] = hX - Xh = \alpha(h)X$, which implies that for an element $X \in \mathfrak{g}_\alpha$ of the form 1, we must have $A = 0$ and $w = -v^T$. Thus,

$$\alpha(h)X = h \cdot \begin{pmatrix} 0 & -v^T & 0 \\ v & 0 & -v \\ 0 & -v^T & 0 \end{pmatrix}.$$

From this calculation, the vector v has $n-1$ dimensions, which implies that the root α has multiplicity $n-1$. Thus, we denote the root decomposition as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, where

$$\mathfrak{g}_0 = \mathfrak{m} \oplus \mathfrak{a}, \quad \mathfrak{g}_\alpha = \left\{ \begin{pmatrix} 0 & v^T & 0 \\ -v & 0 & v \\ 0 & v^T & 0 \end{pmatrix} \mid v \in \mathbb{R}^{n-1} \right\},$$

and the negative root space is

$$\mathfrak{g}_{-\alpha} = \left\{ \begin{pmatrix} 0 & v^T & 0 \\ -v & 0 & -v \\ 0 & -v^T & 0 \end{pmatrix} \mid v \in \mathbb{R}^{n-1} \right\}.$$

Let the nilpotent subalgebra $\mathfrak{n} = \mathfrak{g}_\alpha$, with $\dim(\mathfrak{n}) = n-1$. Then, the Iwasawa decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{so}(n,1)$ is given by

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n},$$

and the corresponding Iwasawa decomposition of the Lie group is $G = KAN$, where $K = SO(n)$.

We define the Killing form as $B(x,y) = \mathrm{Tr}(\mathrm{ad}_x \cdot \mathrm{ad}_y)$ where $x, y \in \mathfrak{g}$. This induces an inner product $\langle x, y \rangle = \frac{1}{2(n-1)} B(x,y)$. Thus, the Killing form B is positive-definite over $\mathfrak{p}$ and is negative-definite $\mathfrak{k}$. The reason for this normalization is so that the Riemannian metric on $G/K = \mathcal{H}^n$ induced by $\langle \cdot, \cdot \rangle$ has a constant curvature -1.

This normalization gives

$$\langle \mathrm{a}_1, \mathrm{a}_1 \rangle = \frac{1}{2(n-1)} B(\mathrm{a}_1, \mathrm{a}_1) = 1.$$

With respect to this inner product, we fix unit root vectors n_i as an orthonormal basis for the subalgebra $\mathfrak{n}$, where $i = 1, \ldots, n-1$, and let $\{m_j\}$ be an orthonormal basis for $\mathfrak{m}$. This gives us an orthonormal basis for $\mathfrak{g}$:

$$\{n_i\} \cup \{\theta(n_i)\} \cup \{m_j\} \cup \mathrm{a}_1.$$

Denoting

$$k_i = \frac{1}{\sqrt{2}}(n_i + \theta(n_i)) \in \mathfrak{k},$$
$$p_i = \frac{1}{\sqrt{2}}(n_i - \theta(n_i)) \in \mathfrak{p},$$

we obtain another orthonormal basis:

$$\{k_i\} \cup \{p_i\} \cup \{m_j\} \cup \mathrm{a}_1, \tag{2}$$

where $\{k_i\} \cup \{m_j\}$ forms an orthonormal basis for $\mathfrak{k}$, and $\{p_i\} \cup \{\mathrm{a}_1\}$, with $1 < i \leq \dim(\mathfrak{p}) = n$, forms an orthonormal basis for $\mathfrak{p}$. Now the Casimir operator with respect to the basis 2 can be written as

$$\Omega = \sum_i -k_i^2 + \sum_j -m_j^2 + \mathrm{a}_1^2 + \sum_i p_i^2.$$

And we have the differential operator $\mathcal{L}_\Omega$ for a smooth function f defined over the Lie group G, if we take $g = kan \in G$ where $a = \exp(t\mathrm{a}_1)$, then by adpoting the theorem 2.1, we have the expression of $\mathcal{L}_\Omega f(g)$ for the rank-one connected Lie group,

$$\begin{aligned} \mathcal{L}_\Omega f(g) = & A_{\mathrm{a}_1}^2 f(g) + (n-1) A_{\mathrm{a}_1} f(g) \\ & + \sum_{s=2}^{n} \left[-2\sqrt{2} e^{-t} K_{k_{s-1}} N_{n_{s-1}} f(g) + 2e^{-2t} N_{n_{s-1}}^2 f(g) \right] \\ & + K_{\Omega_M} f(g). \end{aligned} \tag{3}$$

3.2 Representation theory of orthogonal groups

3.2.1 Representation of K

The representation of the Lie group $SO(n,1)$ is induced by the representation of its subgroup $SO(n-1)$. Therefore, in this section, we provide a summary of the representation theory of $SO(n)$. We are considering the Lie group $G = SO(n,1)$, in this case the maximal compact subgroup $K \cong SO(n)$. And $M = SO(n-1)$ has similar representations as K. The rank of K is p where $n = 2p+1$ or $n = 2p$. And the rank of M is p when $n = 2p+1$ and $p-1$ when $n = 2p$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{k}$,

$$\mathfrak{h} = \left\{ \begin{pmatrix} E_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & E_p \end{pmatrix} \mid E_i = \begin{pmatrix} 0 & x_i \\ -x_i & 0 \end{pmatrix}, x_1, \ldots x_p \in \mathbb{R} \right\}.$$

We extend $< \cdot, \cdot >$ to a Hermitian inner product of $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes \mathbb{C}$:

$$(x + iy, x' + iy') = (x, x') + (y, y').$$

We identify $\mathfrak{h}$ with its dual using this form. Let $(i\epsilon_1, \ldots, i\epsilon_p)$ be an ordered orthonormal basis of $\mathfrak{h}_\mathbb{C}$. Let Σ denote the set of simple roots for $\mathfrak{h} \subset \mathfrak{k}$ corresponding to the lexicographical ordering of $i\mathfrak{h}$. We have

$$\Sigma = \begin{cases} \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{2p-1} - \epsilon_{2p}, \epsilon_{2p}\}, & \text{if } n = 2p+1, \\ \{\epsilon_1 - \epsilon_2, \ldots, \epsilon_{2p-1} - \epsilon_{2p}, \epsilon_{2p-1} + \epsilon_{2p}\}, & \text{if } n = 2p. \end{cases} \tag{4}$$

The equivalence classes of unitary irreducible representations ω of K are parameterized by highest weights:

$$\Lambda_\omega = \sum_{i=1}^{p} \Lambda_{\omega,i} \cdot \epsilon_i,$$

where the highest weights must satisfy the dominance conditions:

$$\frac{2(\Lambda_\omega, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_{\geq 0}, \alpha \in \Sigma.$$

This gives:

* $n = 2p : \Lambda_{\omega,1} \geq \Lambda_{\omega,2} \geq \cdots \geq \Lambda_{\omega,p-1} \geq |\Lambda_{\omega,p}|$,

* $n = 2p+1, \Lambda_{\omega,1} \geq \Lambda_{\omega,2} \geq \cdots \geq \Lambda_{\omega,p-1} \geq \Lambda_{\omega,p} \geq 0$,

these numbers are in $\frac{1}{2}\mathbb{Z}$. The non-integral case corresponds to representations of universal cover $\widetilde{SO}(n)$ that do not factor through $SO(n)$.
The summation of positive roots:

$$2\sigma = \begin{cases} 2[(p-1)\epsilon_1 + (p-2)\epsilon_2 + \cdots + \epsilon_p], & \text{if } n = 2p, \\ (2p-1)\epsilon_1 + (2p-3)\epsilon_2 + \cdots + \epsilon_p, & \text{if } n = 2p+1. \end{cases} \tag{5}$$

For the Casimir operator Ω_K, it acts on irreducible representations as a constant

$$\omega(\Omega_K) = (\Lambda_\omega + \sigma, \Lambda_\omega + \sigma) - (\sigma, \sigma). \tag{6}$$

3.2.2 Representations of $SO(n,1)^\circ$

In this section, we provide a brief overview of the representation theory of the Lorentz group $SO(n,1)^\circ$, summarizing on Thieleker's work [6, 5]. To avoid technical complications, we focus primarily on the identity component.
The special orthogonal group $SO(n,1)$ has two connected components, with the identity component denoted by $SO(n,1)^\circ$. To minimize unnecessary technical difficulties, we will restrict our attention to the representation theory of this identity component. The parabolic subgroup P of G has a decomposition MAN. And we denote the sum of all positive roots for $(\mathfrak{g}, \mathfrak{a})$ as 2ρ.
The representations of G are induced from the representations of M. For an irreducible unitary representation $\mu : M \to V_\mu$ of M, we consider the space of smooth functions

$$V_G = \left\{ F : G \to V_\mu \mid F(xman) = \eta(m) e^{(\nu+\rho)(\log(a))} F(x), \ \|F\| < \infty \right\},$$

where $\nu \in \mathfrak{a}_{\mathbb{C}}^*$ is determined by $\nu(\mathrm{a}_1)$, and $\rho(\mathrm{a}_1) = n - 1$. And the norm is given by

$$\|F\|^2 = \int_K |F(k)|^2 \, dk.$$

We take the completion of this space V_G with the action of G defined by $\pi(g)F(x) = F(g^{-1}x)$. In this way, we get the induced representation (π, V_G, G) associated to (η, ν). If $\nu \in i\mathfrak{a}^*$, this representation is unitary, but not in general. However, in some cases where $\nu \notin i\mathfrak{a}^*$, it can be equipped with a different inner product to make it unitary. Analysis of irreducible components of this representation leads to a complete classification of the unitary dual.

Before the classification of irreducible representation of $G = SO(n, 1)$, let's clear the construction. For a fixed irreducible unitary representation (η, V_η, M) and $\nu \in i\mathfrak{a}_{\mathbb{C}}^*$, we can construct the induced representation $(\pi_{\eta,\nu}, V_G, G)$ associated to (η, ν). By restricting the representation $(\pi_{\eta,\nu}, V_G, G)$ to the maximal compact subgroup K, it can be decomposed into a direct summation

$$\pi_{\eta,\nu} \mid_K = \bigoplus_{\omega \in \widehat{K}} \omega.$$

The following two theorems from Lemma 4 of [5] provide the decomposition of the representation (ω, K) into irreducible representations of (η, M), and describe the relationship between the highest weight irreducible representation (ω, K) and its restriction to (η, M).

Theorem 3.1. *Consider the restriction of the representation π of the Lie group G, to the maximal compact subgroup (ω, K). In this case, each ω appears with multiplicity one in the decomposition*

$$\pi \mid_K = \bigoplus_{\omega \in \widehat{K}} \omega.$$

Furthermore, every irreducible representation (ω, K) can be decomposed into irreducible M-components (η, M) as follows:

$$\omega \mid_M = \bigoplus_{\substack{\Lambda_{\omega,i+1} \geq \Lambda_{\eta,i} \\ \geq \Lambda_{\omega,i}}} \eta.$$

where the highest weight $\Lambda_{\widetilde{\eta}}$ of $\widetilde{\eta}$ is determined by the highest weight Λ_ω of ω. And the equivalent class $[\omega] \in \widehat{K}$ occurs in the restriction (π, G) with the same multiplicity as $[\eta] \in \widehat{M}$.

Theorem 3.2 (Branching law)**.** *Each irreducible representation (ω, K) is restricted from the unitary representation $\pi_{\mu,\nu}$ that is induced from the irreducible representation (μ, M), the highest weight of (ω, K) and (μ, M) satisfies the following condition:*

* *For the even case $n = 2p$, the highest weight holds* $\Lambda_{\omega,1} \geq \Lambda_{\eta,1} \geq \Lambda_{\omega,2} \geq \Lambda_{\eta,2} \geq \cdots \geq \Lambda_{\eta,p-1} \geq |\Lambda_{\omega,p}|$,
* *For the odd case $n = 2p+1$, the highest weight holds* $\Lambda_{\omega,1} \geq \Lambda_{\eta,1} \geq \Lambda_{\omega,2} \geq \Lambda_{\eta,2} \geq \cdots \geq \Lambda_{\eta,p-1} \geq \Lambda_{\omega,p} \geq |\Lambda_{\eta,p}|$.

Extensive research has focused on the spectral properties of the Laplacian on locally symmetric spaces. In the context of this document, we utilize a representation-theoretic variant of Weyl's law about the spectrum properties of the Casimir eigenvalues for real rank one Lie group as shown in the theorem 9.1 of [2].

Theorem 3.3 (Weyl's law [2]). *For the compact homogeneous space $M = \Gamma\backslash G$, the counting function $N(t) = \sum_{\mu\in\Sigma_p,\|\mu\|\leq t} m(\mu)$ for spectrum μ of principal series smaller than t is summed with multiplicities $m(\mu)$*

$$N(t) = (2\sqrt{\pi})^{-n}\Gamma\left(\frac{n}{2}+1\right)\operatorname{vol}(M)t^n + O(t^{n-1}/\log t), \quad t \to +\infty.$$

In general, any unitary representation of a Lie group G is unitarily equivalent to a direct integral of irreducible representations. In the context of our study, the compactness of M implies that the representation (π, H, G) can be decomposed into a Hilbert direct sum of irreducible representations (π_μ, H_μ, G). Each π-invariant subspace is associated with a Casimir eigenvalue μ, and the multiplicities of these eigenvalues are determined according to the version of Weyl's law 3.3 previously referenced.

From the preceding discussion, the Casimir eigenvalue μ of an irreducible spherical unitary representation (π, H, G) emerges as the central parameter for classifying unitary representations. For a unitary representation (π, H, G), the subspaces H_μ and $H_{\mu,\varpi}$ are defined by

$$H_\mu = \{v \in C^\infty(H) \mid \mathcal{L}_\Omega v = \mu v\},$$
$$H_{\mu,\varpi} = \{v \in C^\infty(H_\mu) \mid K_{\Omega_K} v = \varpi v\}.$$

Focusing exclusively on the cocompact lattice Γ, the spectral decomposition of the representation H can be delineated as

$$H = \bigoplus_{\mu\in\Sigma_p\cup\Sigma_f} H_\mu = \bigoplus_{\mu\in\Sigma_p\cup\Sigma_f}\bigoplus_{\varpi} H_{\mu,\varpi}, \tag{7}$$

where $\Sigma_p \cup \Sigma_f \subset \operatorname{Spec}(\pi)$ and $\varpi \in \operatorname{Spec}(\pi|_K)$. Here, Σ_p denotes the discrete part of the spectrum corresponding to the principal series, such that $\Sigma_p \subset -\frac{(n-1)^2}{4} - \mathbb{R}_{\geq 0}$, and Σ_f represents the finite part of the spectrum encompassing the complementary series, the end-point representations, and the trivial representation.

Now in the following subsections, there is a complete list of all the irreducible unitary representations of Lie group G.

3.2.2.1 Principal series representation

Fix a pair (η, ν) where η is a unitary irreducible representation of M and $\nu(\mathrm{a}_1) \in i\mathbb{R}_{\geq 0}$. Since we are working in rank one, we identify $\mathfrak{a}_{\mathbb{C}}^*$ with $\mathbb{C}$ and adopt the convention $\nu := \nu(\mathrm{a}_1)$. Here, the discussion excludes the case where $\nu = 0, n = 2p \geq 4$ with the highest weight $\Lambda_{\eta,p-1}$ is a half odd integer. In the case of the principal series representation, $\nu = is$ with $s \in \mathbb{R}_{\geq 0}$. We take the induced representation $\pi_{s,\eta}$ associated to (η, ν).
For the Casimir eigenvalue μ,

$$\mu = \begin{cases} -s^2 - \frac{(n-1)^2}{4} + \sum\limits_{i=1}^{p-1}(\Lambda_{\eta,i} + n - 2i - 1)\Lambda_{\eta,i}, & \text{if } n = 2p, \\ -s^2 - \frac{(n-1)^2}{4} + \sum\limits_{i=1}^{p}(\Lambda_{\eta,i} + n - 2i - 1)\Lambda_{\eta,i}, & \text{if } n = 2p+1. \end{cases} \tag{8}$$

3.2.2.2 Complementary series representation

Given η an irreducible unitary representation of M, and j the smallest integer with $\Lambda_{\eta,j} = 0$, for the odd case $n = 2p+1$, we assume $\Lambda_{\eta,p} = 0$, such that $\nu \in \left(0, \frac{n+1}{2} - j\right)$, for the even case $n = 2p$: if $\Lambda_{\eta,p-1} \neq 0$, we assume $\Lambda_{\eta,p-1} \in \mathbb{Z}$, then $\nu \in \left(0, \frac{1}{2}\right)$, if $\Lambda_{\eta,p-1} = 0$, $\nu \in \left(0, \frac{n+1}{2} - j\right)$. We take the induced representation $\pi_{\eta,\nu}$ equipped with a suitable inner product.
The Casimir eigenvalues are

$$\mu = \nu^2 - \frac{(n-1)^2}{4} + \sum_{i=1}^{p-1}(\Lambda_{\eta,i} + n - 2i - 1)\Lambda_{\eta,i},\ n > 3.$$

3.2.2.3 End-point representation

Here the pair (η, ν) and the integer j is defined as above. We assume $j \leq p$ for the odd case and $j \leq p - 1$ for the even case. Let $m \in \mathbb{Z}_{\geq 0}$ and $n \leq \Lambda_{\eta,j-1}$, then

$$\nu = \frac{n+1}{2} - j + m.$$

We take the induced representation π_m associated to (η, ν).
First, we consider the subrepresentation π_m^+ associated to $\omega \in \widehat{K}$ with $\Lambda_{\omega,j} > m$ equipped with a suitable inner product. Second, when $m = 0$, we consider the subrepresentation π_0^- associated to $\omega \in \widehat{K}$ with $\Lambda_{\omega,j} = 0$ equipped with a suitable inner product.
The Casimir eigenvalue in both cases:

$$\mu = (m + 1 - j)(m + n - j) + \sum_{i=1}^{j}(\Lambda_{\eta,i} + n - 2i - 1)\Lambda_{\eta,i}.$$

3.2.2.4 Discrete series representation

Same definition of η as above, $\nu = -\frac{1}{2}$ and $\Lambda_{\eta,p-1} \in \mathbb{Z}_{\geq 0}$. We take the induced representation π associated to (η, ν) and the subrepresentation σ corresponding to the summation of $\omega \in \widehat{K}$ with $\Lambda_{\omega,p} = 0$ equipped with a suitable inner product.
The Casimir eigenvalue is

$$\sigma(\Omega) = p(p-1) + \sum_{i=1}^{p-1} (\Lambda_{\eta,i} + n - 2i - 1)\Lambda_{\eta,i}.$$

For the other case that $\Lambda_{\eta,p-1} \neq 0$, we denote $\nu = -s - \frac{1}{2}$. We assume $\nu \geq 0$, and that s is a number such that $-\Lambda_{\eta,p-1} \leq s \leq \Lambda_{\eta,p-1}$ and $s - \Lambda_{\eta,p-1} \in \mathbb{Z}$. Let π be the induced representation associated to (η, ν), the subrepresentation σ_s^+ associated to $\omega \in \widehat{K}$ such that $\Lambda_{\omega,p} \geq -s$, the subrepresentations σ_s^- associated to $\omega \in \widehat{K}$ such that $\Lambda_{\omega,p} \leq s$ equipped with suitable inner products.
The Casimir eigenvalue is

$$\sigma^{\pm}(\Omega) = (s+p)(s-p+1) + \sum_{i=1}^{p-1} (\Lambda_{\eta,i} + n - 2i - 1)\Lambda_{\eta,i}.$$

3.2.2.5 Spherical Representation

In order to construct the representation of G, it needs the irreducible representation (η, M). And as we discussed in the section 1.1, the non-trivial integral $I(a_t, f)$ provides K-invariant vectors. The branching law 3.2 tells us that the K-types appearing in π are determined by the weights of the representation (η, M). In order for the trivial K-type (i.e., the weight-zero component) to occur in $\pi|_K$, the inducing representation (η, M) must itself contribute a weight-zero component.
Since (η, M) is irreducible, this is only possible if (η, M) is trivial. Hence, to ensure that the integral $I(a_t, f)$ produces a nonzero K-invariant vector and that π is spherical, it is necessary that the irreducible representation (η, M) be trivial.

Lemma 2. *The spherical representation of G can be only principal series with $\nu = i\mathbb{R}_{\geq 0}$, complementary series $\nu \in \left(0, \frac{n-1}{2}\right)$ and trivial representation with $\mu = 0$. In both cases of principal series and complementary series, the Casimir eigenvalue $\mu = \nu^2 - \frac{(n-1)^2}{4}$.*

Remark. The proof of this lemma 2 is a direct consequence of the classification of all irreducible representations of G in Thieleker's work [6], which is collected in the section 3.2. And because the representation (η, M) has to be trivial with zero highest weight, then a part of the isolated-point representation and all discrete series are impossible to be spherical.

3.2.3 Harmonic Analysis on the Group $SO(n,1)$

Here we recall the definition of the Sobolev norm, For any $s \in \mathbb{R}_{\geq 0}$, the L^2-Sobolev space $W_s(H)$ of order s is defined to be the domain of the operator $(1+\mathcal{L}_\Delta)^{s/2}$. The space $W_s(H)$ is equipped with an inner product given by

$$\langle \phi, \psi \rangle_{W_s} = \langle (1+\mathcal{L}_\Delta)^s \phi, \psi \rangle, \quad \text{for all } \phi, \psi \in C^\infty(H),$$

where 1 denotes the identity operator on H. This inner product turns the Sobolev space $W_s(H)$ into a Hilbert space, whose associated norm is denoted by $\|\cdot\|_{W_s}$. We define the Sobolev spaces $W_s(H_\nu)$ and $W_s(H_{\nu,\varpi})$ for any $\nu \in \mathrm{Spec}(\pi)$ and $\varpi \in \mathrm{Spec}(\pi|_K)$. Recalling the decomposition of the unitary representation π, it is a fact that the decomposition induces analogous decompositions at the level of Sobolev spaces, namely, there are orthogonal decompositions

$$W_s(H) = \bigoplus_\nu W_s(H_\nu) = \bigoplus_\nu \bigoplus_\varpi W_s(H_{\nu,\varpi}).$$

The Sobolev norm as defined above satisfies the following lemma:

Lemma 3. *Consider an irreducible representation (π_ν, H_ν, G) characterized by the Casimir eigenvalue μ, which is parametrized by ν. For any real number $s \geq 0$, a natural number $k \in \mathbb{N}$, and a vector u in the Sobolev space $W_{s+k}(H_{\nu,\varpi})$, the following relationship between the norms of u in different Sobolev spaces holds:*

$$\|u\|^2_{W_{s+k}} = (1+\mu-2\varpi)^k \|u\|^2_{W_s}.$$

Proof. We may assume without loss of generality that $k=1$. The self-adjoint properties of Lie derivatives with respect to the previously defined inner product are used in the calculation. Consequently, the norm of u in the Sobolev space W_{s+1} can be computed as follows:

$$\begin{aligned}
\|u\|^2_{W_{s+1}} &= \langle u, u \rangle_{W_{s+1}} = \langle (1+\mathcal{L}_\Delta)^{s+1} u, u \rangle \\
&= \langle (1+\mathcal{L}_\Delta)^s u, u \rangle + \langle (1+\mathcal{L}_\Delta)^s u, \mathcal{L}_\Delta u \rangle \\
&= (1+\mu-2\varpi) \langle (1+\mathcal{L}_\Delta)^s u, u \rangle,
\end{aligned}$$

where s is any non-negative real number. This derivation is valid for any natural number k, leading to the general result:

$$\|u\|_{W_{s+k}} = (1+\mu-2\varpi)^k \|u\|_{W_s}.$$

□

By the ellipticity of the operator $\mathcal{L}_\Delta$, we introduce another definition of the Sobolev norm that is equivalent to Definition 1.

Definition 3. If k is an integer, then on the subspace $C^k(H) \subset W_k(H)$, the norm $\|\cdot\|_{W_k}$ is equivalent to the norm $\|\cdot\|_{W'_k}$ defined by

$$\|v\|^2_{W'_k} = \sum \|\pi\left(X_\alpha\right) v\|^2 ,$$

where the sum runs over all monomials $X_\alpha = X_{i_1} X_{i_2} \dots X_{i_l} \in U(\mathfrak{g})$ of degree $\leq k$.

As usual, $C^\infty(H)$ is given the topology induced by all the norms $\|\cdot\|_{W_k}$ with $k \in \mathbb{Z}^+$. This makes $C^\infty(H)$ into a Fréchet space; see Lemma 1.6.4 of [7] for details.

4 Reduction to An Ordinary Differential Equations

This section elucidates the essence of the strategy we adopt to prove the theorem related to the asymptotic behavior of the spherical average of eigenfunctions of the Casimir operator $\mathcal{L}_\Omega$. We demonstrate that the eigenvalue equations, which are partial differential equations defining the eigenfunctions, yield an ordinary differential equation for the corresponding K-averages $I(a_t, f)$, when these averages are considered as functions of the parameter t.
Consider a function $f : G \to H$ associated with an irreducible unitary spherical representation (π, H, G), where f is an eigenfunction of the Casimir operator $\mathcal{L}_\Omega$, i.e., $\mathcal{L}_\Omega f = \mu f$. For instance, we define $f(g) = \pi(g)v$ for a fixed, arbitrary vector $v \in H$. This function f is used to define the spherical average $I(a_t, f)$. To simplify notation, we omit the explicit dependence on v. To simplify notation and avoid redundancy, the fixed initial vector v will be omitted from the notation for the function f.

Lemma 4. *Let (π, H, G) be an irreducible unitary representation, and define the function $f : G \to H$ by $f(g) = \pi(g)v$ for a fixed vector $v \in H$. Then f is an eigenfunction of the Casimir operator $\mathcal{L}_\Omega$.*

Remark. the proof of this lemma is immediate from the fact that the Casimir operator is a central element in the universal enveloping algebra of the Lie algebra, and as such it commutes with the action of any element from the Lie group via the representation π. For the details about this property of the Casimir operator, we refer to the theorem 8.6 of the book [3].

Given an eigenfunction f as above, we define the derivatives of the function $I(a_t, f)$ as follows:

$$I'(a_t, f) = \frac{\partial}{\partial t} I\left(a_t, f\right) = \int_K A_{a_1} f(a_t k)\, dk,$$
$$I''(a_t, f) = \frac{\partial^2}{\partial t^2} I(a_t, f) = \int_K A_{a_1}^2 f(a_t k)\, dk.$$

Proposition 1. *For an irreducible unitary spherical representation (π, H, G), let μ be a Casimir eigenvalue of the Casimir operator $\mathcal{L}_\Omega$ associated with the representation π. Consider any eigenfunction $f : G \to H$ that satisfies the equation $\mathcal{L}_\Omega f = \mu f$. There exists a bounded continuous function $G_f(t) : \mathbb{R}_{>0} \to H$ such that the spherical average $I(a_t, f) = \int_K f(a_t k)\, dk$ satisfies the linear inhomogeneous ordinary differential equation*

$$I''(a_t, f) + (n-1)I'(a_t, f) - \mu I(a_t, f) = 2e^{-2t} G_f(t)$$

for all $t > 0$.

Proof. Substituting the differential operator $\mathcal{L}_\Omega f$ into $I(a_t, \mathcal{L}_\Omega f)$, we obtain the equality

$$I(a_t, \mathcal{L}_\Omega f) = I(a_t, \mu f).$$

We aim to analyze the spherical average $I(a_t, f)$, which maps G to H and exhibits K-invariance. Since $M \subseteq K$, the average is also M-invariant by inclusion. Adopting the expression 3 for $\mathcal{L}_\Omega$, the integral $I(a_t, \mathcal{L}_\Omega f)$ simplifies as follows:

$$\begin{aligned} I(a_t, \mathcal{L}_\Omega f) &= \int_K A_{\mathrm{a}_1}^2 f(a_t k)\, dk + (n-1) \int_K A_{\mathrm{a}_1} f(a_t k)\, dk + 2e^{-2t} \sum_{s=2}^{n} \int N_{n_{s-1}}^2 f(a_t k)\, dk. \\ &= I''(a_t, f) + (n-1)I'(a_t, f) - 2e^{-2t} G_f(t), \end{aligned}$$

where

$$G_f(t) = -\sum_{s=2}^{n} \int_K N_{n_{s-1}}^2 f(a_t k)\, dk = -\sum_{s=2}^{n} I(a_t, N_{n_{s-1}}^2 f). \tag{9}$$

Consequently, we deduce the complete ordinary differential equation

$$I''(a_t, f) + (n-1)I'(a_t, f) - \mu I(a_t, f) = 2e^{-2t} G_f(t).$$

□

5 Asymptotic Analysis of Eigenfunctions

This section aims to derive an explicit asymptotic formula for various irreducible unitary spherical representations by addressing the ordinary differential equations presented in the proposition 1. Our approach begins with the resolution of a general Cauchy problem, subject to initial conditions specified at $t = 1$.

5.1 Solutions of Ordinary Differential Equation

Lemma 5. *If $G : \mathbb{R}_{\geq 0} \to H$ is a continuous vector-valued function where H is a Hilbert space, then for any vectors $y_1, y_2 \in H$ and constant $\mu \in \mathbb{C}$, the solution to the Cauchy problem*

$$\begin{cases} y''(t) + (n-1)y'(t) - \pi(\Omega)y(t) = 2e^{-2t}G(t), \\ y(1) = y_1, \\ y'(1) = y_2 \end{cases}$$

is given by the following formulae. If $\mu \neq -\frac{(n-1)^2}{4}$, then the solution is given by

$$\begin{aligned} y(t) &= e^{\lambda_- t}C_1 + e^{\lambda_+ t}C_2 \\ &\quad - \frac{1}{\nu}\left(e^{\lambda_- t}\int_1^t e^{\xi(\lambda_- + n - 3)}G(\xi)\,d\xi - e^{\lambda_+ t}\int_1^t e^{\xi(\lambda_+ + n-3)}G(\xi)\,d\xi\right). \end{aligned}$$

where the constants are

$$C_1 = \frac{\lambda_+ y_1 - y_2}{2\nu e^{\lambda_-}}, \quad C_2 = \frac{y_2 - \lambda_- y_1}{2\nu e^{\lambda_+}}. \tag{10}$$

If $\mu = -\frac{(n-1)^2}{4}$, the solution is

$$\begin{aligned} y(t) &= e^{\frac{1}{2}(1-n)t}C_1 + C_2 t e^{\frac{1}{2}(1-n)t} \\ &\quad + 2e^{\frac{1}{2}(1-n)t}\left(t\int_1^t e^{\frac{1}{2}(n-5)\xi}G(\xi)\,d\xi - \int_1^t e^{\frac{1}{2}(n-5)\xi}G(\xi)\xi\,d\xi\right). \end{aligned}$$

with constants

$$C_1 = e^{\frac{1}{2}(n-1)}\left(y_2 + \frac{1}{2}(n-1)y_1\right), \quad C_2 = e^{\frac{1}{2}(n-1)}\left(\frac{1}{2}(3-n)y_1 - y_2\right). \tag{11}$$

Proof. The characteristic polynomial of the homogeneous ordinary differential equation 5 $y''(t) + (n-1)y'(t) - \pi(\Omega)\,y(t) = 0$ is $X^2 + (n-1)X - \mu = 0$, which has discriminant

$$\mathcal{D} = \frac{(n-1)^2}{4} + \mu. \tag{12}$$

It has two distinct roots $\lambda_\pm = \frac{1-n}{2} \pm \nu$ if $\mathcal{D} \neq 0$ and a double root $\frac{1-n}{2}$ if $\mathcal{D} = 0$. The property of Casimir eigenvalues μ control the property of roots.

If there are two distinct roots, a general solution of the inhomogeneous ordinary equation is given as

$$y(t) = e^{\lambda_- t}C_1 + e^{\lambda_+}C_2 - \frac{1}{\nu}\left(e^{\lambda_- t}\int_1^t e^{\xi(\lambda_- + n-3)}G(\xi)\,d\xi - e^{\lambda_+ t}\int_1^t e^{\xi(\lambda_+ + n-3)}G(\xi)\,d\xi\right),$$

which can be easily verified by computations. Imposing in the initial conditions $y(1) = y_1$ and $y'(1) = y_2$ enables to determine the coefficients

$$C_1 = \frac{\lambda_+ y_1 - y_2}{2\nu e^{\lambda_-}}, C_2 = \frac{y_2 - \lambda_- y_1}{2\nu e^{\lambda_+}}.$$

By some minor modifications, we can obtain the solutions for the case of a double root:

$$y(t) = e^{\frac{1}{2}(1-n)t}C_1 + C_2 t e^{\frac{1}{2}(1-n)t} + 2e^{\frac{1}{2}(1-n)t}\left(t\int_1^t e^{\frac{1}{2}(n-5)\xi}G(\xi)\,d\xi - \int_1^t e^{\frac{1}{2}(n-5)\xi}G(\xi)\xi\,d\xi\right).$$

with constants

$$C_1 = e^{\frac{1}{2}(n-1)}\left(y_2 + \frac{1}{2}(n-1)y_1\right), C_2 = e^{\frac{1}{2}(n-1)}\left(\frac{1}{2}(3-n)y_1 - y_2\right).$$

□

Now we can apply this lemma 5 to the proposition 1 to obtain the explicit analytic expression of the average integral $I(a_t, f)$ with the coefficients $C_\pm(\mathcal{D}, f)$ and the function $G_f(t)$, and the coefficients $C_\pm(\mathcal{D}, f)$ are defined as follows: and note that the coefficients $C_\pm(\mathcal{D}, f)$ are vectors in Hilbert space H with the irreducible representation (π, H, G) which has suitable inner product to keep it unitary. When the representation is fixed in the discussion, we shorten the notation $C_\pm(\mathcal{D}, f)$ as $C_\pm(f)$.

Corollary 1. *The average $I(a_t, f)$ satisfies the proposition 1 and can be expressed as the solution to the ordinary differential equation*

$$\begin{aligned} I(a_t, f) &= e^{\lambda_- t}C_+(\nu, f) + e^{\lambda_+ t}C_-(\nu, f) \\ &\quad - \frac{1}{\nu}\left(e^{\lambda_- t}\int_1^t e^{\xi(\lambda_+ + n - 3)}G_f(\xi)\,d\xi - e^{\lambda_+ t}\int_1^t e^{\xi(\lambda_- + n - 3)}G_f(\xi)\,d\xi\right). \end{aligned} \tag{13}$$

when $\mathcal{D} \neq 0$; and

$$\begin{aligned} I(a_t, f) &= e^{\frac{1}{2}(1-n)t}C_-(0, f) + C_+(0, f)te^{\frac{1}{2}(1-n)t} \\ &\quad + 2e^{\frac{1}{2}(1-n)t}\left(t\int_1^t e^{\frac{1}{2}(n-5)\xi}G_f(\xi)\,d\xi - \int_1^t e^{\frac{1}{2}(n-5)\xi}G_f(\xi)\xi\,d\xi\right). \end{aligned} \tag{14}$$

when $\mathcal{D} = 0$.

And the constants $C_\pm(\mathcal{D}, f)$ satisfies the following expression and conditions,

1. *if $\mathcal{D} \neq 0$,*

$$C_\pm(\nu, f) = \frac{\pm\lambda_\pm \int_K f(k)\,dk \mp \int_K A_{\mathrm{a}_1} f(k)\,dk}{2\nu e^{\lambda_\mp}} \tag{15}$$

And it holds that

$$||C_\pm(\nu, f)|| \leq |\frac{\lambda_\pm + 1}{2\nu e^{\lambda_\mp}}|||f||_{W_1}. \tag{16}$$

2. *if $\mathcal{D} = 0$,*

$$\begin{aligned} C_+(\mathcal{D}, f) &= e^{\frac{n-1}{2}} \left(\int_K A_{\mathrm{a}_1} f(k)\, dk + \frac{n-1}{2} \int_K f(k)\, dk \right), \\ C_-(\mathcal{D}, f) &= e^{\frac{n-1}{2}} \left(\frac{3-n}{2} \int_K f(k)\, dk - \int_K A_{\mathrm{a}_1} f(k)\, dk \right). \end{aligned} \tag{17}$$

And it holds that

$$||C_\pm(\mathcal{D}, f)|| \le \frac{1}{2}(1 \pm n) e^{\frac{1}{2}(n-1)} ||f||_{W_1}. \tag{18}$$

5.2 Upper Bound of the Spherical Average $I(a_t, f)$

In this section, we want to use the expression 1 of the solutions of ordinary differential equation 5 to obtain an explicit upper bound of the average $I(a_t, f)$. We note that in the expression 13, the integrand's exponent, $\xi(n-3+\lambda_\pm)$, incorporates the positive coefficient $n-3$ for ξ, which introduces the divergence in the integral. And it holds the same issue when $D = 0$. Within the definition of G_f, the expressions $\int_K N_{n_{s-1}}^2 f(a_t k)\, dk$ are observed, both of which comply with the solutions to the corresponding ordinary differential equation. These expressions may be iteratively substituted into G_f to generate a series of approximations until a convergent integral is obtained. Therefore, in the following proposition 2, we establish upper bounds for $||I(a_t, f)||$, which are analyzed in relation to the classification of irreducible spherical representations and the discriminant $\mathcal{D}$. Before stating the proposition, we classify the discriminant $\mathcal{D}$ using the lemma 2 on the classification of irreducible spherical representations.

Corollary 2. *The discriminant $\mathcal{D}$ of the ordinary differential equation 5, associated with the Casimir eigenvalue μ in an irreducible spherical unitary representation, is given by:*

$$\mathcal{D} \in \begin{cases} \mathbb{R}_{\le 0}, & \text{Principal Series,} \\ \left(0, \frac{(n-1)^2}{4}\right), & \text{Complementary Series,} \\ \frac{(n-1)^2}{4}, & \text{End-point Representation.} \end{cases}$$

Proposition 2. *Given a non-trivial irreducible representation (π, H, G), the Casimir eigenvalue associated with it is given by $\mu = \nu^2 - \frac{(n-1)^2}{4}$.*

1. *For the principal series with $\nu \in i\mathbb{R}_{>0}$, the norm of the integral $I(a, f)$ satisfies*

$$\|I(a_t, f)\| \ll_n e^{\frac{1}{2}(1-n)t} \|f\|_{W_1} \max\left(1, \frac{1}{|\nu|}\right) + e^{-2t} \max_{i_1} \|I(a_t, N_{i_1}^2 f)\|.$$

2. *For the principal series with $\nu = 0$, it satisfies*

$$||I(a_t, f)|| \ll_n e^{\frac{1}{2}(1-n)t} t ||f||_{W_1} + t e^{-2t} \max_{i_1} ||I(a_t, N_{i_1}^2 f)||.$$

3. For the complementary series with $\nu \in \left(0, \frac{n-1}{2}\right)$, it satisfies

$$||I(a_t, f)|| \ll_n e^{\left(\frac{1}{2}(1-n)+\nu\right)t}||f||_{W_1} \max\left(1, \frac{1}{|\nu|}\right) + e^{-2t} \max_{i_1}||I(a_t, N_{i_1}^2 f)||,$$

where the implied constant in $\ll_n$ depends on the fixed parameter n.

Proof. Let us revisit the expression for

$$G_f(t) = \sum_{s=2}^{n} \int_K N_{n_{s-1}}^2 f(ak)\, dk = \sum_{s=2}^{n} I(a_t, N_{n_{s-1}}^2 f).$$

To simplify the notation, the index i_1 corresponds to a vector from the basis $\{n_{s-1}\}$. As t approaches infinity, the vector norm of $G_f(t)$ can be estimated by

$$\|G_f(t)\| \ll_n \max_{i_1}\|I(a_t, N_{i_1}^2 f)\|.$$

Here, the term $I(a, N_{i_1}^2 f)$ remains bounded by a constant multiplied by $\|f\|_{W_3}$, due to the compactness of the subgroup K.
Now, turning our attention to the case where $\mathcal{D} \neq 0$, we consider the vector norm of $I(a_t, f)$. Applying the triangle inequality yields

$$\begin{aligned}\|I(a, f)\| \leq e^{\frac{1}{2}(1-n)t}&\left(e^{-\mathrm{Re}(\nu)t}\|C_+(\mathcal{D}, f)\| + e^{\mathrm{Re}(\nu)t}\|C_-(\mathcal{D}, f)\|\right) \\ &+ \frac{1}{|\nu|}e^{\frac{1}{2}(1-n)t}\left(e^{-\mathrm{Re}(\nu)t}\left\|\int_1^t e^{(n-3+\lambda_+)\xi}G_f(\xi)\, d\xi\right\|\right. \\ &\left.+ e^{\mathrm{Re}(\nu)t}\left\|\int_1^t e^{(n-3+\lambda_-)\xi}G_f(\xi)\, d\xi\right\|\right).\end{aligned}$$

Invoking the trivial bound for $G_f(t)$, the integral terms can be calculated as

$$\begin{aligned}\|I(a, f)\| \leq e^{\frac{1}{2}(1-n)t}&\left(e^{-\mathrm{Re}(\nu)t}\|C_+(\mathcal{D}, f)\| + e^{\mathrm{Re}(\nu)t}\|C_-(\mathcal{D}, f)\|\right) \\ &+ \frac{1}{\|\nu\|}e^{\frac{1}{2}(1-n)t}\left(e^{-\mathrm{Re}(\nu)t}\, e^{(n-3+\mathrm{Re}(\lambda_+))t} \max_{i_1}\|I(a_t, N_{i_1}^2 f)\|\right. \\ &\left.+ e^{\mathrm{Re}(\nu)t}\, e^{(n-3+\mathrm{Re}(\lambda_-))t} \max_{i_1}\|I(a_t, N_{i_1}^2 f)\|\right).\end{aligned}$$

Utilizing the bounds for the constants $C_+(\mathcal{D}, f)$ and $C_-(\mathcal{D}, f)$, we note that

$$\begin{aligned}e^{-\mathrm{Re}(\nu)t}\|C_+(\mathcal{D}, f)\| + e^{\mathrm{Re}(\nu)t}\|C_-(\mathcal{D}, f)\| &\ll_n e^{\mathrm{Re}(\nu)t}\|f\|_{W_1}\frac{1}{|\nu|}\sqrt{\left(\frac{3-n}{2}\right)^2 + |\nu|^2}, \\ &\ll_n e^{Re(\nu)t}\, \|f\|_{W_1} \max\left(1, \frac{1}{|\nu|}\right).\end{aligned}$$

Then the estimate for $\|I(a_t, f)\|$ simplifies to

$$\|I(a_t, f)\| \ll_n e^{\frac{1}{2}(1-n)t} e^{\mathrm{Re}(\nu)t} \|f\|_{W_1} \max\left(1, \frac{1}{|\nu|}\right) + e^{-2t} \max_{i_1} \|I(a_t, N_{i_1}^2 f)\|.$$

Consider the scenario where $\mathcal{D} = 0$. In this particular case, the solution can be expressed as

$$I(a_t, f) = e^{\frac{1}{2}(1-n)t} \Bigg\{ C_-(0, f) + C_+(0, f)t + 2\left[t \int_1^t e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi - \int_1^t e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, \xi\, d\xi \right] \Bigg\}.$$

Upon examining the vector norm of $I(a_t, f)$, we deduce that

$$\|I(a_t, f)\| \le e^{\frac{1}{2}(1-n)t} \Big(\|C_-(0, f)\| + \|C_+(0, f)\|\, t \Big) + 2e^{\frac{1}{2}(1-n)t} \left[t \left\| \int_1^t e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi \right\| + \left\| \int_1^t e^{\frac{1}{2}(n-5)\xi}\, \xi\, G_f(\xi)\, d\xi \right\| \right].$$

Next, invoking the trivial bound for $G_f(\xi)$, we obtain

$$\begin{aligned}
&\le e^{\frac{1}{2}(1-n)t} \left(\|C_-(0, f)\| + \|C_+(0, f)\| t \right) \\
&\quad + e^{\frac{1}{2}(1-n)t} \left(t \left\| \int_1^t e^{\frac{1}{2}(n-5)\xi} \max_{i_1} I(a_\xi, N_{i_1}^2 f)\, d\xi \right\| + \left\| \int_1^t e^{\frac{1}{2}(n-5)\xi} \max_{i_1} I(a_\xi, N_{i_1}^2 f) \xi\, d\xi \right\| \right) \\
&\ll_n e^{\frac{1}{2}(1-n)t} \left(\|C_-(0, f)\| + \|C_+(0, f)\| t \right) + t e^{-2t} \max_{i_1} \|I(a_t, N_i^2 f)\|.
\end{aligned}$$

and by using the upper bound of the constant $C_\pm(0, f)$, we have the final result

$$\|I(a_t, f)\| \ll_n e^{\frac{1}{2}(1-n)t} t \, \|f\|_{W_1} + t e^{-2t} \|I(a_t, N_i^2 f)\|.$$

$\square$

Lemma 6. *Given an integer $k = \left\lfloor \frac{n-1}{4} \right\rfloor + 1$, the integral $I(a_t, f)$ satisfies the following nontrivial upper bound.*

1. *For the principal series with $\nu \in i\mathbb{R}_{>0}$,*

$$\|I(a_t, f)\| \ll_n \max\left(1, \frac{1}{|\nu|}\right) e^{\frac{1}{2}(1-n)t} \|f\|_{W_{2k+1}}. \tag{19}$$

2. *For the principal series with $\nu = 0$,*

$$\|I(a_t, f)\| \ll_n e^{\frac{1}{2}(1-n)t} t \|f\|_{W_{2k+1}}. \tag{20}$$

3. *For the complementary series with $\nu \in \left(0, \frac{n-1}{2}\right)$,*

$$\|I(a_t, f)\| \ll_n \max\left(1, \frac{1}{|\nu|}\right) e^{\left(\frac{1}{2}(1-n)+\nu\right)t} \|f\|_{W_{2k+1}}. \tag{21}$$

Proof. Recall the notation i_1 denotes a vector from the basis $\{n_{s-1}\}$. We observe that $I(a_t, N_{i_1}^2 f)$ satisfies the differential equation 13 and adheres to the estimate from the previous formula **??**. Substituting the bound of $I(a_t, N_{i_1}^2 f)$ into the inequality **??**, we obtain:

$$\begin{aligned}\|I(a_t,f)\| &\ll_n \max\left(1,\frac{1}{|\nu|}\right) e^{\frac{1}{2}(1-n)t} e^{\mathrm{Re}(\nu)t}\|f\|_{W_1} + e^{-2t}\max_{i_1}\|I(a_t,N_{i_1}^2 f)\|,\\ &\ll_n \max\left(1,\frac{1}{|\nu|}\right) e^{\frac{1}{2}(1-n)t} e^{\mathrm{Re}(\nu)t}\|f\|_{W_1}\\ &\quad + e^{-2t}\left(e^{\frac{1}{2}(1-n)t}e^{\mathrm{Re}(\nu)t}\|f\|_{W_3}\max\left(1,\frac{1}{|\nu|}\right) + e^{-2t}\max_{i_1,i_2}\|I(a_t,N_{i_2}^2N_{i_1}^2 f)\|\right),\\ &\ll_n e^{\frac{1}{2}(1-n)t}e^{\mathrm{Re}(\nu)t}\max\left(1,\frac{1}{|\nu|}\right)\left(\|f\|_{W_1}+\|f\|_{W_3}e^{-2t}\right)\\ &\quad + e^{-4t}\max_{i_1,i_2}\|I(a_t,N_{i_2}^2N_{i_1}^2 f)\|.\end{aligned}$$

After iterating $k = \left\lfloor \frac{n-1}{4} \right\rfloor + 1$ times, this yields

$$\begin{aligned}\|I(a_t,f)\| &\ll_n \max\left(1,\frac{1}{|\nu|}\right) e^{\frac{1}{2}(1-n)t} e^{\mathrm{Re}(\nu)t}\|f\|_{W_{2k+1}} + e^{-2kt}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|,\\ &\ll_n \max\left(1,\frac{1}{|\nu|}\right) e^{\frac{1}{2}(1-n)t}\left(e^{\mathrm{Re}(\nu)t}\|f\|_{W_{2k+1}} + e^{\left(\frac{1}{2}(n-1)-2k\right)t}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|\right).\end{aligned}$$

The second term, $e^{-2kt}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|$, satisfies

$$e^{\left(\frac{1}{2}(n-1)-2k\right)t}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|\| = O(1).$$

where the integral part remains bounded.

The method is also applied in the case when $\mathcal{D}=0$, ensuring that the integral $I(a_t,f)$ satisfies the estimate:

$$\begin{aligned}\|I(a_t,f)\| &\ll_n e^{\frac{1}{2}(1-n)t}t\|f\|_{W_1} + te^{-2t}\max_{i_1}\|I(a_t,N_{i_1}^2 f)\|,\\ &\ll_n e^{\frac{1}{2}(1-n)t}t\left(\|f\|_{W_1}+te^{-2t}\|f\|_{W_3}\right) + t^2e^{-4t}\max_{i_1,i_2}\|I(a_t,N_{i_2}^2N_{i_1}^2 f)\|.\end{aligned}$$

Continuing the iteration for $k = \left\lfloor \frac{n-1}{4} \right\rfloor + 1$ times, we obtain:

$$\begin{aligned}\|I(a_t,f)\| &\ll_n e^{\frac{1}{2}(1-n)t}t\|f\|_{W_{2k+1}} + t^ke^{-2kt}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|,\\ &\ll_n e^{\frac{1}{2}(1-n)t}\left(t\|f\|_{W_{2k+1}} + t^{2k}e^{\left(\frac{1}{2}(n-1)-2k\right)t}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|\right).\end{aligned}$$

The second term, $t^ke^{-2kt}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\|$, satisfies

$$t^{2k}e^{\left(\frac{1}{2}(n-1)-2k\right)t}\max_{i_1,\dots,i_k}\|I(a_t,N_{i_k}^2\dots N_{i_1}^2 f)\| = O(1).$$

where the integral part remains bounded. □

As a direct consequence of the iteration lemma 6 above, we have a non-trivial bound for the function $G_f(t)$:

Corollary 3. *Recall the definition 9 of the function*

$$G_f(t) = -\sum_{s=2}^{n} I(a_t, N^2_{n_{s-1}} f) \ll_n \max_{i_1} ||I(a_t, N^2_{i_1} f),$$

which has the following upper bounds:

1. *For the principal series with* $\nu \in i\mathbb{R}_{>0}$*,*

$$\|G_f(t)\| \ll_n \max\left(1, \frac{1}{|\nu|}\right) e^{\frac{1}{2}(1-n)t} \|f\|_{W_{2k+3}}. \tag{22}$$

2. *For the principal series with* $\nu = 0$*,*

$$\|G_f(t)\| \ll_n e^{\frac{1}{2}(1-n)t} t \|f\|_{W_{2k+3}}. \tag{23}$$

3. *For the complementary series with* $\nu \in \left(0, \frac{n-1}{2}\right)$*,*

$$\|G_f(t)\| \ll_n \max\left(1, \frac{1}{\nu}\right) e^{\left(\frac{1}{2}(1-n)+\nu\right)t} \|f\|_{W_{2k+3}}. \tag{24}$$

5.3 Asymptotic formula for irreducible spherical representations

The goal of this section is to establish asymptotic formulae for all possible irreducible unitary representations. And as the discussion in the section 1.1, the non-trival spherical average $I(a_t, f)$ provides K-invariant vectors. So we start this section by specify the discriminant with all possible spherical irreducible representations, which is an easy consequence of the lemma 2. Then we will discuss each case explicitly to establish asymptotic formulae.

Lemma 7. *The discriminant* $\mathcal{D}$ *of the ordinary differential equation, associated with the Casimir eigenvalue* $\mu = \nu^2 - \frac{(n-1)^2}{4}$ *in an irreducible spherical unitary representation, is given by:*

$$\mathcal{D} \in \begin{cases} \mathbb{R}_{\leq 0}, & \textit{Principal Series}, \\ \left(0, \frac{(n-1)^2}{4}\right), & \textit{Complementary Series}. \end{cases}$$

Now combining the iteration lemma 6 and the upper bound of G_f in the corollary 3, we can obtain explicit asymptotic formulae for each case of irreducible representations.

Theorem 5.1. *Let μ be the Casimir eigenvalue of irreducible spherical unitary representation (π, H) of the group G. And let $k = \left[\frac{n-1}{4}\right] + 1$, suppose $f \in W_{2k+5}(G)$ satisfies that $\mathcal{L}_\Omega f = \mu f$. And define the spherical average $I(a_t, f) = \int_K f(a_t k)\, dk$.*

1. *For the trivial representation, i.e. $\mu = 0$,*

$$I(a_t, f) = \int_K f(k)\, dk.$$

2. *For the principal series representation, i.e., $\mu = \nu^2 - \frac{(n-1)^2}{4}$ where $\nu \in i\mathbb{R}_{\geq 0}$, we consider two cases:*

   (a) *if $\nu \in i\mathbb{R}_{>0}$, there exist two vectors $\Theta^\pm \in H$ of finite norm, bounded by the Sobolev norm of the function f, i.e.,*

$$\left\|\Theta^\pm\right\| \ll_n \max\left(1, \frac{1}{|\nu|^2}\right) \|f\|_{W_{2k+3}}.$$

   Thus, for $t \geq 1$, we have

$$I(a_t, f) = e^{\lambda_- t}\Theta^- + e^{\lambda_+ t}\Theta^+ + \mathcal{R}(f, t),$$

   where the error term satisfies

$$\|\mathcal{R}(f, t)\| \ll_n \max\left(1, \frac{1}{|\nu|^2}\right) \|f\|_{W_{2k+3}} e^{-\frac{1}{2}(n+3)t}.$$

   (b) *If $\nu = 0$, all the coefficients satisfy the bounds*

$$\left\|\Theta^\pm(f)\right\| \ll_n \|f\|_{W_{2k+3}}, \quad \left\|\Theta_t^{(1)}\right\| \ll_n \|f\|_{W_{2k+5}}, \quad \left\|\Theta_{t^2}^{(1)}\right\| \ll \|f\|_{W_{2k+5}}.$$

   Thus, for $t \geq 1$, we obtain the asymptotic formula:

$$\begin{aligned} I\left(a_t, f\right) = {} & e^{\frac{1}{2}(1-n)t}\Theta^-(f) + te^{\frac{1}{2}(1-n)t}\Theta^+(f) \\ & + e^{-\frac{1}{2}(3+n)t}(1+4t)\Theta_t^{(1)} \\ & + e^{-\frac{1}{2}(3+n)t}(4t^2+3t+1)\Theta_{t^2}^{(1)} \\ & + \mathcal{R}(f, t)^{(1)}. \end{aligned}$$

   where the error term satisfies

$$\left\|\mathcal{R}(f, t)^{(1)}\right\| \ll_n e^{-\frac{1}{2}(7+n)t} t^3 \|f\|_{W_{2k+5}}.$$

3. *For the complementary series representation, i.e., $\mu = \nu^2 - \frac{(n-1)^2}{4}$ where $\nu \in \left(0, \frac{n-1}{2}\right)$, we need to discuss three different cases separately, considering the different relationships between ν and integers.*

(a) *For complementary series representation with $\nu \in (0,1)$, Given two polynomials $P^{\pm}(\nu)$ and vectors $\Theta^{\pm}(f)$, as well as the sum $\sum_{s=2}^{n} \Theta^{\pm}(N_{n_{s-1}}^2 f)$, which satisfy*

$$\left\|\Theta^{\pm}(f)\right\| \ll_\nu \|f\|_{W_{2k+3}}, \quad \left\|\Theta_{\pm,-2}^{(1)}\right\| \ll_\nu \|f\|_{W_{2k+5}}$$

where the $\ll_\nu$ depends on the parameter ν. we obtain, for $t \geq 1$,

$$I(a_t, f) = e^{\lambda_- t}\Theta^-(f) + e^{\lambda_+ t}\Theta^+(f) + e^{(\lambda_- -2)t}\Theta_{-,-2}^{(1)}(f) + e^{(\lambda_+ -2)t}\Theta_{+,-2}^{(1)}(f) + \mathcal{R}(f,t)^{(1)}$$

The error term satisfies the bound

$$\left\|\mathcal{R}(f,t)^{(1)}\right\| \ll_n \frac{1}{\nu^3(1-\nu)} e^{\left(-\frac{1}{2}(7+n)+\nu\right)t} \|f\|_{W_{2k+5}}.$$

(b) *For complementary series representation with $\nu = 1$, similarly, there exists vectors $\Theta_+^{(2)}(f)$, $\Theta_{-,t}^2(f)$ and $\Theta_-^{(2)}(f)$ satisfying*

$$\left\|\Theta_{\pm}^{(2)}(f)\right\| \ll_n \|f\|_{W_{2k+7}}, \quad \left\|\Theta_{-,t}^2(f)\right\| \ll_n \|f\|_{W_{2k+7}},$$

where $\ll_n$ depends only on the fixed integer n. For $t \geq 1$, the asymptotic formula is given by

$$\begin{aligned} I(a_t, f) = {} & e^{\lambda_- t}\Theta_-^{(2)}(f) + e^{\lambda_+ t}\Theta_+^{(2)}(f) - e^{\lambda_- t}(t-1)\,\Theta_{-,t}^2(f) \\ & + \mathcal{R}(f,t)^{(2)} \end{aligned}$$

where the error term satisfies

$$\left\|\mathcal{R}(f,t)^{(2)}\right\| \ll_n e^{-\frac{1}{2}(3+n)t} \|f\|_{W_{2k+7}}.$$

(c) *For the complementary series representation with $\nu \in (1,2)$, there exist many vectors $\Theta^+(f)$ and $\Theta_{+,-2}^{(1)}(f)$ that satisfy*

$$\left\|\Theta^+(f)\right\| \ll_\nu \|f\|_{W_{2k+3}}, \quad \left\|\Theta_{+,-2}^{(1)}(f)\right\| \ll_\nu \|f\|_{W_{2k+5}}.$$

Here, the implicit constant $\ll_\nu$ depends only on the eigenvalue parameter ν. For $t \geq 1$, the asymptotic formula is given by

$$I(a_t, f) = e^{\lambda_+ t}\Theta^+(f) - e^{(\lambda_+ -2)t}\Theta_{+,-2}^{(1)}(f) + \mathcal{R}'(f,t)^{(1)},$$

where the error term satisfies the bound

$$\left\|\mathcal{R}'(f,t)^{(1)}\right\| \ll_\nu e^{(-\frac{1}{2}(n+3)+\nu-2)t} \|f\|_{W_{2k+5}}.$$

(d) *For complementary series representation with $\nu = 2$, similarly, There are many vectors $\Theta^{(2)}_{+,-2i}(f)$ with $i = 0, 1, 2$, $\Theta^{(2)}_{-,t}(f)$ and $\Theta^{(2)}_{-}(f)$ that satisfies*

$$\left\|\Theta^{(2)}_{+,-2i}(f)\right\| \ll_n \|f\|_{W_{2k+7}}, \left\|\Theta^{(2)}_{-,t}(f)\right\| \ll_n \|f\|_{W_{2k+7}}, \left\|\Theta^{(2)}_{-}(f)\right\| \ll_n \|f\|_{W_{2k+7}}$$

where $\ll_n$ depends only on the fixed integer n. For $t \geq 1$, the asymptotic formula is given by

$$I(a_t, f) = \sum_{i=0}^{2} e^{(\lambda_+ - 2i)t}\Theta^{(2)}_{+,-2i}(f) + e^{\lambda_- t}\Theta^{(2)}_{-}(f) \quad + e^{\lambda_- t}(t-1)\,\Theta^{(2)}_{-,t} + \mathcal{R}(f,t)^{(2)}$$

where the error term satisfies

$$\left\|\mathcal{R}(f,t)^{(2)}\right\| \ll_n e^{-\frac{1}{2}(3+n)t}\|f\|_{W_{2k+7}}.$$

(e) *For a complementary series representation with $\nu \in \left(2, \frac{n-1}{2}\right)$ and an integer $l = \left\lfloor \frac{\nu}{2} \right\rfloor + 1$, the spherical average $I(a_t, f)$ is defined by the function $f \in W_{2k+3+2l}(G)$. And the asymptotic formula of $I(a_t, f)$ takes the form*

$$I(a_t, f) = \sum_{i=0}^{l} e^{(\lambda_+ - 2i)t}\Theta^{(l)}_{+,-2i}(f) + \mathcal{R}'(f,t)^{(l)}.$$

For any $i = 0, \dots, l$, the vector $\Theta^{(l)}_{+,-2i}(f)$ satisfies

$$\left\|\Theta^{(l)}_{+,-2i}(f)\right\| \ll_\nu \|f\|_{W_{2k+3+2i}},$$

Then, the error term $\mathcal{R}'(f,t)^{(l)}$ satisfies

$$\|\mathcal{R}'(f,t)\| \ll_\nu e^{(\lambda_+ - 2l)t}\|f\|_{W_{2k+3+2l}}.$$

where $\ll_\nu$ only depends on the parameter ν.

Remark. For any ν corresponding to any different irreducible representation, the error term $\mathcal{R}(f,t)$ exhibits a decay rate of at least $e^{-\frac{1}{2}(n+3)t}$. As the iteration process continues, the decay rate improves progressively, leading to even faster convergence.

For the sequence of derivatives, we introduce the shorthand notation

$$\sum_{s_1=2}^{n}\sum_{s_2=2}^{n}\cdots\sum_{s_l=2}^{n} N^2_{n_{s_1-1}} N^2_{n_{s_2-1}} \cdots N^2_{n_{s_l-1}} f = \sum_{\vec{s}} N^{(l)}_{\vec{s}} f,$$

where $\vec{s} = (s_1, s_2, \dots, s_l)$ is a multi-index with $s_i \in \{2, \dots, n\}$ for each i, and

$$N^{(l)}_{\vec{s}} f = N^2_{n_{s_1-1}} N^2_{n_{s_2-1}} \cdots N^2_{n_{s_l-1}} f.$$

5.3.1 Trivial Representation

Consider a specific function $f(g) = \pi(g)v$, where π is the trivial representation. In this context, the Casimir eigenvalue associated with the trivial representation, μ, is zero. Consequently, we obtain

$$I(a_t, f) = \int_K f(k)\, dk = v.$$

For the spherical average, we require that the irreducible representation be spherical. According to a result by Thieleker [6], this condition implies that the Casimir eigenvalue μ vanishes, yielding two distinct roots for the characteristic equation:

$$\lambda_- = 1 - n, \quad \lambda_+ = 0.$$

In this case, the representation (π, H, G) is induced by the trivial representation M, and (π, H) is a one-dimensional representation on the space of constant functions. Therefore, the spherical end-point representation is the trivial representation.
Consequently, the solution to the ordinary differential equation under consideration takes the form

$$I(a_t, f) = C_-\left(\frac{(n-1)^2}{4}, f\right) + C_+\left(\frac{(n-1)^2}{4}, f\right) e^{(1-n)t}$$

and the formula for the constants

$$\begin{aligned} C_+\left(\frac{(n-1)^2}{4}, f\right) &= -\frac{e^{n-1}}{n-1} \int_K A_{\mathfrak{a}_1} f(k)\, dk = 0, \\ C_-\left(\frac{(n-1)^2}{4}, f\right) &= \int_K f(k)\, dk + \frac{1}{n-1} \int_K A_{\mathfrak{a}_1} f(k)\, dk. \end{aligned}$$

Therefore, the spherical average $I(a_t, f)$ is just the $K-$invariant vector, i.e.

$$I(a_t, f) = \int_K f(k)\, dk = v.$$

5.3.2 Principal Series Representation with $\nu = 0$

The first case is when $\mathcal{D} = 0$, i.e. $\nu = 0$. In this situation, we recall the solution of the corresponding ordinary differential equation:

$$\begin{aligned} I(a_t, f) = {} & e^{\frac{1}{2}(1-n)t} C_-(0, f) + C_+(0, f)\, t\, e^{\frac{1}{2}(1-n)t} \\ & + 2e^{\frac{1}{2}(1-n)t} \left(t \int_1^t e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi - \int_1^t e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, \xi\, d\xi \right). \end{aligned}$$

This expression can be rewritten as

$$I(a_t, f) = e^{\frac{1}{2}(1-n)t} \Theta^-_{p_0}(f) + t\, e^{\frac{1}{2}(1-n)t} \Theta^+_{p_0}(f) + \mathcal{R}_{p_0}(f, t),$$

where the coefficients are defined by

$$\Theta_{p_0}^{\pm}(f) = C_{\pm}(0, f) \pm 2\int_1^{\infty} e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi,$$

and the residual term is given by

$$\begin{aligned}\mathcal{R}_{p_0}(f, t) = 2e^{\frac{1}{2}(1-n)t} &\int_t^{\infty} \xi\, e^{\frac{1}{2}(n-5)\xi}\, G_f(\xi)\, d\xi \\ - 2t\, e^{\frac{1}{2}(1-n)t} &\int_t^{\infty} e^{\frac{1}{2}(n-5)\xi}\, G_f(\xi)\, d\xi.\end{aligned}$$

By using an appropriate upper bound for the constant $C_{\pm}(0, f)$, we obtain

$$\begin{aligned}\left\|\Theta_{p_0}^{\pm}\right\| &\leq \|C_{\pm}(0, f)\| \pm 2\left\|\int_1^{\infty} e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi\right\|, \\ &\ll_n \|C_{\pm}(0, f)\| \pm \|f\|_{W_{2k+3}} \int_1^{\infty} e^{-2\xi}\xi\, d\xi, \\ &\ll_n \|f\|_{W_{2k+3}}.\end{aligned}$$

Similarly, the residual term $\mathcal{R}_{p_0}(f, t)$ satisfies

$$\begin{aligned}\|\mathcal{R}_{p_0}(f, t)\| \leq 2e^{\frac{1}{2}(1-n)t} &\left(\left\|\int_t^{\infty} \xi\, e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi\right\|\right. \\ &\left. + \; t\left\|\int_t^{\infty} e^{\frac{1}{2}(n-5)\xi} G_f(\xi)\, d\xi\right\|\right), \\ &\ll_n e^{\frac{1}{2}(1-n)t}\, \|f\|_{W_{2k+3}} \left(\int_t^{\infty} \xi^2 e^{-2\xi}\, d\xi + t\int_t^{\infty} \xi\, e^{-2\xi}\, d\xi\right), \\ &\ll_n t^2 e^{-\frac{1}{2}(3+n)t}\, \|f\|_{W_{2k+3}}.\end{aligned}$$

Before proceeding with further calculations, we introduce some abbreviated notation. For $l \in \mathbb{N}_{\geq 1}$, which represents the number of iterations, we define a sequence of integrals as follows:

$$J_{1,p_0} = \int_t^{\infty} (\xi_1 - t)e^{-2\xi_1}\, d\xi_1, \qquad J_{l,p_0} = \int_{\xi_{l-1}}^{\infty} (\xi_l - \xi_{l-1})e^{-2\xi_l}\, d\xi_l.$$

For simplicity, we also note that

$$\int_t^{\infty} (\xi_1 - t)e^{-2\xi_1} \left(\int_{\xi_1}^{\infty} (\xi_2 - \xi_1)e^{-2\xi_2}\, d\xi_2\right) d\xi_1 = J_{1,p_0}(t)\, J_{2,p_0}(\xi_1).$$

The error term corresponding to the original expression for G_f can be written as

$$\begin{aligned}\mathcal{R}_{p_0}(f, t) = 2\sum_{s=2}^{n} &\left(e^{\frac{1}{2}(1-n)t} \int_t^{\infty} \xi\, e^{\frac{1}{2}(n-5)\xi_1} I(a_{\xi_1}, N_{n_{s-1}}^2 f)\, d\xi_1\right. \\ &\left. - \; t\, e^{\frac{1}{2}(1-n)t} \int_t^{\infty} e^{\frac{1}{2}(n-5)\xi_1} I(a_{\xi_1}, N_{n_{s-1}}^2 f)\, d\xi_1\right).\end{aligned}$$

Since $I(a_{\xi_1}, N^2_{n_{s-1}} f)$ satisfies a similar asymptotic formula, an iterative process is initiated. Each time the asymptotic formula for $I(a_t, N^2_{n_{s-1}} f)$ is expanded, higher orders of derivatives in the N-component emerge.
We then define two integrals as follows:

$$J^{\Theta}_{l,p_0}(\xi_{l-1}) = \int_{\xi_{l-1}}^{\infty} (\xi_l - \xi_{l-1}) e^{-2\xi_l} \Big(\Theta^-_{p_0}\big(N^{(l)}_{\vec{s}} f\big) + \xi_l \, \Theta^+_{p_0}\big(N^{(l)}_{\vec{s}} f\big)\Big) d\xi_l,$$
$$J^{G}_{l,p_0}(\xi_{l-1}) = \int_{\xi_{l-1}}^{\infty} \Big(\xi_l - \xi_{l-1}\Big) \, e^{\frac{1}{2}(n-5)\xi_l} G_{N^{(l-1)}_{\vec{s}} f}(\xi_l) \, d\xi_l.$$

For each iteration, the main term is given by

$$\varphi_{l,p_0} = e^{\frac{1}{2}(1-n)t} e^{-2lt} \Theta_{l+1}(f,t),$$

where $\Theta_{l+1}(f,t)$ is a polynomial in t with coefficients that depend on the Sobolev norm of f of at most degree l. It can be written implicitly as

$$\Theta_{l+1}(f,t) = P^-_l(t) \sum_{\vec{s}} \Theta^-_{p_0}\big(N^{(l)}_{\vec{s}} f\big) + P^+_{l+1}(t) \sum_{\vec{s}} \Theta^+_{p_0}\big(N^{(l)}_{\vec{s}} f\big),$$

where $P^{\pm}_l(t)$ is a polynomial in t with real coefficients of at most degree l. Furthermore, by a calculation analogous to that for $\Theta^{\pm}_{p_0}(f)$, one obtains

$$\left\| \sum_{\vec{s}} \Theta^{\pm}_{p_0}\big(N^{(l)}_{\vec{s}} f\big) \right\| \ll_n \|f\|_{W_{2k+2l+3}}.$$

In the case $l = 0$, the notation φ_{0,p_0} coincides with the original formula,

$$\varphi_{0,p_0} \triangleq e^{\frac{1}{2}(1-n)t} \, \Theta_0(f,t),$$

with Í$\Theta_0(f,t) = \Theta^-(f) + t\,\Theta^+(f)$. After l iterations, the error term can be expressed as

$$\mathcal{R}_{l,p_0}(f,t) = -(-2)^{l+1} e^{\frac{1}{2}(1-n)t} \sum_{\vec{s}} J_{1,p_0}(t) J_{2,p_0}(\xi_1) \cdots J_{l,p_0}(\xi_{l-1}) \cdot J^G_{l+1,p_0}(\xi_l),$$

and it satisfies

$$\|\mathcal{R}_{l,p_0}\| \ll_n e^{\frac{1}{2}(1-n)t} e^{-2lt} Q_{l+2}(t) \, \|f\|_{W_{2k+2l+3}},$$

where $Q_{l+2}(t)$ is a polynomial of t with real coefficients of degree $l+2$.
Thus, we obtain the asymptotic formula

$$\begin{aligned} I(a_t, f) &= \sum_l \varphi_{l,p_0}(f,t) + \mathcal{R}_{l,p_0}(f,t) \\ &= e^{\frac{1}{2}(1-n)t} \sum_l e^{-2lt} \Theta_l(f,t) + \mathcal{R}_{l,p_0}(f,t). \end{aligned}$$

5.3.3 Principal Series Representation with $\nu \neq 0$

First, recall the expression 13 of the average $I(a_t, f)$ when $\mathcal{D} \neq 0$:

$$\begin{aligned} I(a_t, f) &= e^{\lambda_- t} C_+(\mathcal{D}, f) + e^{\lambda_+} C_-(\mathcal{D}, f) \\ &\quad - \frac{1}{\nu}\left(e^{\lambda_- t} \int_1^t e^{\xi(\lambda_- + n - 3)} G_f(\xi)\, d\xi - e^{\lambda_+ t} \int_1^t e^{\xi(\lambda_+ + n - 3)} G_f(\xi)\, d\xi \right). \end{aligned}$$

We rewrite this expression as the following asymptotic formula:

$$I(a_t, f) = e^{\lambda_- t}\Theta^- + e^{\lambda_+ t}\Theta^+ + \mathcal{R}(f, t),$$

where we define the constants $\Theta^\pm$ as

$$\Theta^\pm = C_\mp(\mathcal{D}, f) \pm \frac{1}{\mathcal{D}} \int_1^\infty e^{(n-3+\lambda_\mp)\xi} G_f(\xi)\, d\xi.$$

The term $\mathcal{R}(f, t)$ is a bounded function given by

$$\mathcal{R}(f, t) = e^{\lambda_- t} \frac{1}{\nu} \int_t^\infty e^{(n-3+\lambda_+)\xi} G_f(\xi)\, d\xi - e^{\lambda_+ t} \frac{1}{\nu} \int_t^\infty e^{(n-3+\lambda_-)\xi} G_f(\xi)\, d\xi.$$

These definitions encapsulate the long-term behavior of $I(a_t, f)$, distinguishing the main growth components, $\Theta^\pm(f)$, from the residual term, $\mathcal{R}(f, t)$.
When ν is completely imaginary, then by the upper bound of G_f in the corollary 3, the constants $\Theta^\pm$ can be bounded as follows:

$$\begin{aligned} \|\Theta^\pm\| &\leq \|C_\mp\| + \frac{1}{|\nu|} \int_1^\infty \left| e^{(n-3+\lambda_\mp)\xi} \right| \|G_f(\xi)\|\ d\xi \\ &\ll_n \|C_\mp\| + \frac{1}{|\nu|} \max\left(1, \frac{1}{|\nu|}\right) \int_1^\infty e^{-2\xi}\, d\xi\, \|f\|_{W_{2k+3}}. \end{aligned}$$

Recalling the bound 16 of the constant $\|C_\pm\|$, the upper bound of $\Theta^\pm$ simplifies to

$$\left\|\Theta^\pm\right\| \ll_n \max\left(1, \frac{1}{|\nu|^2}\right) \|f\|_{W_{2k+3}}.$$

And by similar computation, the residual term satisfies

$$\begin{aligned} ||\mathcal{R}(f, t)|| \leq& \frac{1}{|\nu|}\left(\left\| e^{\lambda_- t} \int_t^\infty e^{(n-3+\lambda_+)\xi} G_f(\xi)\, d\xi \right\| + \left\| e^{\lambda_+ t} \int_t^\infty e^{(n-3+\lambda_-)\xi} G_f(\xi)\, d\xi \right\| \right) \\ \ll_n& \frac{1}{|\nu|} \max\left(1, \frac{1}{|\nu|}\right) ||f||_{W_{2k+3}} e^{\frac{1}{2}(1-n)t} \left[\left| \int_t^\infty |e^{(-2+\nu)\xi}|\, d\xi \right| + \left| \int_t^\infty |e^{(-2-\nu)\xi}|\, d\xi \right| \right] \\ \ll_n& \max\left(1, \frac{1}{|\nu|^2}\right) ||f||_{W_{2k+3}} e^{-\frac{1}{2}(n+3)t} \end{aligned}$$

5.3.4 Complementary Series Representation with $\nu \in (0,1)$

In this case of complementary series, the determinant $\mathcal{D}$ is strictly positive, ensuring that the characteristic polynomial has two distinct negative roots. Since the discriminant satisfies $\mathcal{D} \in \left(0, \frac{1}{2}(n-1)\right)$, the roots $\lambda_\pm$ are given by

$$\lambda_+ = \frac{1}{2}(1-n) + \nu, \quad \lambda_- = \frac{1}{2}(1-n) - \nu,$$

with the following bounds:

$$\lambda_+ \in \left(\frac{1}{2}(1-n), 0\right), \quad \lambda_- \in \left(1-n, \frac{1}{2}(1-n)\right).$$

Recall the expression of the integral $I(a_t, f)$ from the solution of ordinary differential equation 13,

$$\begin{aligned} I(a_t, f) &= e^{\lambda_- t} C_+(\nu, f) + e^{\lambda_+ t} C_-(\nu, f) \\ &\quad - \frac{1}{\nu}\left(e^{\lambda_- t}\int_1^t e^{\xi(\lambda_+ + n - 3)} G_f(\xi)\, d\xi - e^{\lambda_+ t}\int_1^t e^{\xi(\lambda_- + n-3)} G_f(\xi)\, d\xi\right). \end{aligned}$$

We rewrite it into an asymptotic formula:

$$I(a_t, f) = e^{\lambda_- t}\Theta^-(f) + e^{\lambda_+ t}\Theta^+(f) + \mathcal{R}(f,t), \tag{25}$$

where we define the constants $\Theta^\pm$ as

$$\Theta^\pm(f) = C_\mp(\nu, f) \pm \frac{1}{\nu}\int_1^\infty e^{(n-3+\lambda_\mp)\xi} G_f(\xi)\, d\xi.$$

The term $\mathcal{R}(f,t)$ is a bounded function given by

$$\mathcal{R}(f,t) = e^{\lambda_- t}\frac{1}{\nu}\int_t^\infty e^{(n-3+\lambda_+)\xi} G_f(\xi)\, d\xi - e^{\lambda_+ t}\frac{1}{\nu}\int_t^\infty e^{(n-3+\lambda_-)\xi} G_f(\xi)\, d\xi.$$

First, we establish an upper bound for $\Theta^\pm(f)$ using the upper bound from (16) for the constants $C_\pm(\nu, f)$ and the corollary 3 about the upper bound of G_f. We derive:

$$\begin{aligned} \left\|\Theta^\pm\right\| &\le \|C_\mp(\nu, f)\| + \frac{1}{\nu}\int_1^\infty e^{(n-3+\lambda_\mp)\xi}\, \|G_f(\xi)\|\, d\xi \\ &\ll_n \|C_\mp(\nu, f)\| + \frac{1}{\nu^2}\|f\|_{W_{2k+3}} \\ &\ll_n \frac{1}{\nu^2}\, \|f\|_{W_{2k+3}} \end{aligned}$$

Next, we estimate $\mathcal{R}(f,t)$ using Corollary 3:

$$\begin{aligned} \|\mathcal{R}(f,t)\| &\le \frac{1}{\nu}\left(e^{\lambda_- t}\int_t^\infty e^{(n-3+\lambda_+)\xi}\, \|G_f(\xi)\|\, d\xi + e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi}\, \|G_f(\xi)\|\, d\xi\right) \\ &\ll_n \frac{1}{\nu^2}\, \|f\|_{W_{2k+3}}\left(e^{\lambda_- t}\left|\int_t^\infty e^{(2\nu-2)\xi}\, d\xi\right| + e^{\lambda_+ t}\left|\int_t^\infty e^{-2\xi}\, d\xi\right|\right) \\ &\ll_n \frac{1}{\nu^2(1-\nu)}\, \|f\|_{W_{2k+3}}\, e^{(\lambda_+ - 2)t} \end{aligned}$$

To obtain a more precise asymptotic formula for $\nu \in (0,1)$, we plug the original definition of the function $G_f(\xi)$ into the residual term $\mathcal{R}(f,t)$:

$$\begin{aligned}\mathcal{R}(f,t) = \frac{1}{\nu}\Bigg(& e^{\lambda_- t}\int_t^\infty e^{\left(n-3+\lambda_+\right)\xi}\sum_{s=2}^{n} I\left(a_\xi, N_{n_{s-1}}^2 f\right)d\xi \\ & - e^{\lambda_+ t}\int_t^\infty e^{\left(n-3+\lambda_-\right)\xi}\sum_{s=2}^{n} I\left(a_\xi, N_{n_{s-1}}^2 f\right)d\xi\Bigg).\end{aligned}$$

We note that the integral $I(a_\xi, N_{n_{s-1}}^2 f)$ satisfies the asymptotic formula **??**. By substituting this asymptotic behavior into $\mathcal{R}(f,t)$, we obtain

$$\begin{aligned}\mathcal{R}(f,t) = \frac{1}{\nu}\sum_{s=2}^{n}\Bigg[& e^{\lambda_- t}\int_t^\infty e^{\left(n-3+\lambda_+\right)\xi}\left(e^{\lambda_-\xi}\Theta^-\left(N_{n_{s-1}}^2 f\right) + e^{\lambda_+\xi}\Theta^+\left(N_{n_{s-1}}^2 f\right)\right)d\xi \\ & - e^{\lambda_+ t}\int_t^\infty e^{\left(n-3+\lambda_-\right)\xi}\left(e^{\lambda_-\xi}\Theta^-\left(N_{n_{s-1}}^2 f\right) + e^{\lambda_+\xi}\Theta^+\left(N_{n_{s-1}}^2 f\right)\right)d\xi\Bigg] \\ & + \mathcal{R}(f,t)^{(1)}. \end{aligned} \tag{26}$$

Here, the error term $\mathcal{R}(f,t)^{(1)}$ after the first time iteration is given by

$$\begin{aligned}\mathcal{R}(f,t)^{(1)} = -\frac{1}{\nu}\sum_{s=2}^{n}\Bigg[& - e^{\lambda_+ t}\int_t^\infty e^{\left(n-3+\lambda_-\right)\xi}\mathcal{R}\left(N_{n_{s-1}}^2 f, \xi\right)d\xi \\ & + e^{\lambda_- t}\int_t^\infty e^{\left(n-3+\lambda_+\right)\xi}\mathcal{R}\left(N_{n_{s-1}}^2 f, \xi\right)d\xi\Bigg].\end{aligned}$$

Next, in order to shorten notation, denote two vectors by

$$\begin{aligned}\Theta_{-,-2}^{(1)}(f) &= \sum_{s=2}^{n}\Theta^-(N_{n_{s-1}}^2 f)\frac{1}{2\nu}\left(1 - \frac{1}{\nu+1}\right), \\ \Theta_{+,-2}^{(1)}(f) &= \sum_{s=2}^{n}\Theta^+(N_{n_{s-1}}^2 f)\frac{1}{2\nu}\left(\frac{1}{\nu-1} - 1\right)\end{aligned}$$

it becomes

$$\mathcal{R}(f,t) = e^{(\lambda_- -2)t}\Theta_{-,-2}^{(1)}(f) + e^{(\lambda_+ -2)t}\Theta_{+,-2}^{(1)}(f) + \mathcal{R}(f,t)^{(1)}$$

Finally, assembling everything together and plugging into **??**, we obtain the asymptotic formula after the first time iteration:

$$I(a_t, f) = e^{\lambda_- t}\Theta^-(f) + e^{\lambda_+ t}\Theta^+(f) + e^{(\lambda_- -2)t}\Theta_{-,-2}^{(1)}(f) + e^{(\lambda_+ -2)t}\Theta_{+,-2}^{(1)}(f) + \mathcal{R}(f,t)^{(1)}$$

And by using the known upper bounds on $\mathcal{R}(\cdot,\xi)$, we have

$$\left\|\mathcal{R}(f,t)^{(1)}\right\| \ll_n \frac{1}{\nu^3(1-\nu)}e^{(\lambda_+ -4)t}\left\|f\right\|_{W_{2k+5}}.$$

5.3.5 Complementary Series Representation with $\nu \in [1, \frac{n-1}{2})$

For $\nu \geq 1$, recall the solution of the ordinary differential equation 13,

$$\begin{aligned} I(a_t, f) &= e^{\lambda_- t} C_+(\mathcal{D}, f) + e^{\lambda_+ t} C_-(\mathcal{D}, f) \\ &\quad - \frac{1}{\sqrt{\mathcal{D}}}\left(e^{\lambda_- t} \int_1^t e^{\xi(\lambda_- + n - 3)} G_f(\xi)\, d\xi - e^{\lambda_+ t} \int_1^t e^{\xi(\lambda_+ + n-3)} G_f(\xi)\, d\xi \right). \end{aligned}$$

And we rewrite the asymptotic formula in a different way,

$$I(a_t, f) = e^{\lambda_- t}\Theta^-(f) + e^{\lambda_+ t}\Theta^+(f) + \mathcal{R}(f, t),$$

where the constants

$$\begin{cases} \Theta^-(f) = C_+(\nu, f), \\ \Theta^+(f) = C_-(\nu, f) + \frac{1}{\nu}\int_1^\infty e^{(n-3+\lambda_-)\xi} G_f(\xi)\, d\xi \end{cases}$$

and the residual term

$$\mathcal{R}(f, t) = -\frac{1}{\nu}\left(e^{\lambda_- t} \int_1^t e^{(n-3+\lambda_+)\xi} G_f(\xi)\, d\xi + e^{\lambda_+ t} \int_t^\infty e^{(n-3+\lambda_-)\xi} G_f(\xi)\, d\xi \right).$$

We can obtain the same upper bound for the vector $\Theta^+(f)$, and the upper bound for $\Theta^-(f)$ is known from 16. Here, by similar calucation, we obtain the upper bound for the constant vectors:

$$\begin{cases} \|\Theta^-(f)\| \ll_n \|f\|_{W_1}, \\ \|\Theta^+(f)\| \ll_n \left(1 + \frac{1}{\nu}\right) \|f\|_{W_{2k+3}} \end{cases}$$

Next, we estimate the residual term:

$$\begin{aligned} \|\mathcal{R}(f, t)\| &\leq \frac{1}{\nu}\left(e^{\lambda_- t} \int_1^t e^{(n-3+\lambda_+)\xi} \|G_f(\xi)\|\, d\xi + e^{\lambda_+ t} \int_t^\infty e^{(n-3+\lambda_-)\xi} \|G_f(\xi)\|\, d\xi \right) \\ &\ll_n \frac{1}{\nu^2} \|f\|_{W_{2k+3}} \left(e^{\lambda_- t} \int_1^t e^{(2\nu - 2)\xi}\, d\xi + e^{\lambda_+ t} \int_t^\infty e^{-2\xi}\, d\xi \right). \end{aligned}$$

For $\nu = 1$, the residual term satisfies

$$\|\mathcal{R}(f, t)\| \ll_n \|f\|_{W_{2k+3}} e^{-\frac{1}{2}(1+n)t} t.$$

For $\nu > 1$, we obtain

$$\|\mathcal{R}(f, t)\| \ll_n \frac{1}{\nu^2} \|f\|_{W_{2k+3}} e^{(\lambda_+ - 2)t}.$$

The next step is to iterate the asymptotic formula to obtain a more precise error term. We rewrite the term $\mathcal{R}(f, t)$ using the original definition of $G_f(t)$:

$$\mathcal{R}(f,t) = -\frac{1}{\nu}\sum_{s=2}^{n}\Bigg(e^{\lambda_- t}\int_1^t e^{(n-3+\lambda_+)\xi} I(a_\xi, N_{n_{s-1}}^2 f)\, d\xi$$
$$+ e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi} I(a_\xi, N_{n_{s-1}}^2 f)\, d\xi\Bigg).$$

where the integral $I(a_\xi, N_{n_{s-1}}^2 f)$ satisfies the asymptotic formula. Plugging it into $\mathcal{R}(f,t)$, we obtain

$$\begin{aligned}\mathcal{R}(f,t) = &-\frac{1}{\nu}\sum_{s=2}^{n} e^{\lambda_- t}\left[\int_1^t e^{(n-3+\lambda_+)\xi}\big(e^{\lambda_-\xi}\Theta^-(N_{n_{s-1}}^2 f) + e^{\lambda_+\xi}\Theta^+(N_{n_{s-1}}^2 f)\big)\, d\xi\right]\\ &-\frac{1}{\nu}\sum_{s=2}^{n} e^{\lambda_+ t}\left[\int_t^\infty e^{(n-3+\lambda_-)\xi}\big(e^{\lambda_-\xi}\Theta^-(N_{n_{s-1}}^2 f) + e^{\lambda_+\xi}\Theta^+(N_{n_{s-1}}^2 f)\big)\, d\xi\right]\\ &+ \mathcal{R}(f,t)^{(1)},\end{aligned}$$

where the error term after the first iteration is given by

$$\mathcal{R}(f,t)^{(1)} = -\frac{1}{\nu}\sum_{s=2}^{n}\Bigg[e^{\lambda_- t}\int_1^t e^{(n-3+\lambda_+)\xi}\mathcal{R}(N_{n_{s-1}}^2 f, \xi)\, d\xi$$
$$+ e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi}\mathcal{R}(N_{n_{s-1}}^2 f, \xi)\, d\xi\Bigg].$$

Now let's calculate all the integrals above, we obtain the asymptotic formula as

$$\begin{aligned}I(a_t, f) =& e^{\lambda_+ t}\Theta^+(f) + e^{(\lambda_- -2)t}\frac{1}{2(1+\nu)}\sum_{s=2}^{n}\Theta^-(N_{n_{s-1}}^2 f)\\ &+ e^{\lambda_- t}\left(\frac{1}{\nu e^2}\sum_{s=2}^{n}\Theta^-(N_{n_{s-1}}^2 f) + \Theta^-(f)\right)\\ &- \delta_1 e^{\lambda_- t}\frac{1}{\nu}\sum_{s=2}^{n}\Theta^+(N_{n_{s-1}}^2 f) - e^{(\lambda_+ -2)t}\frac{1}{2\nu}\sum_{s=2}^{n}\Theta^+(N_{n_{s-1}}^2 f) + \mathcal{R}(f,t)^{(1)}\end{aligned}$$

where the term δ_m is defined for $m \in \mathbb{Z} \cap [1, \frac{n-1}{2})$

$$\delta_m = \int_1^t e^{(2\nu-2m)\xi}\, d\xi = \begin{cases}\frac{1}{2\nu-2m}\left(e^{(2\nu-2m)t} - e^{2\nu-2m}\right), & \text{if } \nu \neq m,\\ t-1, & \text{if } \nu = m.\end{cases}$$

In order to shorten the notation, we rewrite the formula as

$$\begin{aligned}I(a_t, f) = &e^{\lambda_+ t}\Theta_+^{(0)} + e^{\lambda_- t}\Theta_-^{(0)} + e^{(\lambda_+ -2)t}\Theta_+^{(1)} + e^{(\lambda_- -2)t}\Theta_-^{(1)}\\ &- \delta_1 e^{\lambda_- t}\Theta_1 + \mathcal{R}(f,t)^{(1)}.\end{aligned}$$

where the coefficients $\Theta_{\pm}^{(i)}$ with $i = 0, 1$ and the coefficient Θ_1 are defined as

$$\begin{cases} \Theta_{+}^{(0)} = \Theta^{+}(f), \Theta_{-}^{(0)} = \left(\frac{1}{\nu e^2} \sum\limits_{s=2}^{n} \Theta^{-}(N_{n_{s-1}}^2 f) + \Theta^{-}(f) \right), \\ \Theta_{+}^{(1)} = \frac{1}{2\nu} \sum\limits_{s=2}^{n} \Theta^{+}(N_{n_{s-1}}^2 f), \Theta_{-}^{(1)} = \frac{1}{2(1+\nu)} \sum\limits_{s=2}^{n} \Theta^{-}(N_{n_{s-1}}^2 f), \\ \Theta_1 = \frac{1}{\nu} \sum\limits_{s=2}^{n} \Theta^{+}(N_{n_{s-1}}^2 f). \end{cases}$$

Besides, we note that the sum of the constants satisfies the estimate

$$\left\| \sum_{s=2}^{n} \Theta^{\pm}(N_{n_{s-1}}^2 f) \right\| \ll_{\nu} \|f\|_{W_{2k+5}}.$$

For $\nu = 1$, the asymptotic formula of $I(a_t, f)$ after the first iteration becomes

$$I(a_t, f) = e^{\lambda_+ t} \Theta^{+}(f) - (t-1) e^{\lambda_- t} \sum_{s=2}^{n} \Theta^{+}(N_{n_{s-1}}^2 f) + \mathcal{R}'(f,t)^{(1)},$$

where the error term

$$\mathcal{R}'(f,t)^{(1)} = e^{(\lambda_+ - 2)t} \Theta_{+}^{(1)}(f) + e^{\lambda_- t} \Theta_{-}^{(0)}(f) + e^{(\lambda_- - 2)t} \Theta_{-}^{(1)}(f) + \mathcal{R}(f,t)$$

and the error term satisfies

$$\begin{aligned} \left\| \mathcal{R}'(f,t)^{(1)} \right\| \leq & \sum_{s=2}^{n} \Bigg(e^{\lambda_- t} \int_{1}^{t} e^{(n-3+\lambda_+)\xi} \left\| \mathcal{R}(N_{n_{s-1}}^2 f, \xi) \right\| \, d\xi \\ & + e^{\lambda_+ t} \int_{t}^{\infty} e^{(n-3+\lambda_-)\xi} \left\| \mathcal{R}(N_{n_{s-1}}^2 f, \xi) \right\| \, d\xi \Bigg) \\ & + \left\| e^{\lambda_- t} \left(\Theta^{-}(f) + \frac{1}{e^2} \sum_{s=2}^{n} \Theta^{-}(N_{n_{s-1}}^2 f) + \frac{1}{2} \sum_{s=2}^{n} \Theta^{+}(N_{n_{s-1}}^2 f) \right) \right\| \\ & + \left\| e^{(\lambda_- - 2)t} \left(\frac{1}{4} \sum_{s=2}^{n} \Theta^{-}(N_{n_{s-1}}^2 f) \right) \right\| \\ \ll_n & \left(e^{\lambda_- t} \int_{1}^{t} e^{-2\xi} \xi \, d\xi + e^{\lambda_+ t} \int_{t}^{\infty} e^{-4\xi} \xi \, d\xi \right) \sum_{s=2}^{n} \left\| N_{n_{s-1}}^2 f \right\|_{W_{2k+3}} \\ & + e^{\lambda_- t} \sum_{s=2}^{n} \left\| N_{n_{s-1}}^2 f \right\|_{W_{2k+3}} \\ \ll_n & \, e^{\lambda_- t} \|f\|_{W_{2k+5}}. \end{aligned}$$

Then, we substitute the asymptotic formula back into the original expression of $I(a_t, f)$:

$$\begin{aligned}I(a_t,f) =& e^{\lambda_- t}\Theta^-(f) + e^{\lambda_+ t}\Theta^+(f)\\ &-\sum_{s=2}^{n} e^{\lambda_- t}\int_1^t e^{(n-3+\lambda_+)\xi}\big(e^{\lambda_+\xi}\Theta_+^{(0)}(N_{n_{s-1}}^2 f) - (t-1)e^{\lambda_-\xi}\Theta_1(N_{n_{s-1}}^2 f)\big)\,d\xi\\ &-\sum_{s=2}^{n} e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi}\big(e^{\lambda_+\xi}\Theta_+^{(0)}(N_{n_{s-1}}^2 f) - (t-1)e^{\lambda_-\xi}\Theta_1(N_{n_{s-1}}^2 f)\big)\,d\xi\\ &-\sum_{s=2}^{n} e^{\lambda_- t}\int_1^\infty e^{(n-3+\lambda_+)\xi}\mathcal{R}'\left(N_{n_{s-1}}^2 f,\xi\right)^{(1)}\,d\xi\\ &+\mathcal{R}\left(f,t\right)^{(2)},\end{aligned}$$

where, after the second iteration, the error term becomes

$$\begin{aligned}\mathcal{R}\left(f,t\right)^{(2)} =& \sum_{s=2}^{n}\Bigg(e^{\lambda_- t}\int_t^\infty e^{(n-3+\lambda_+)\xi}\mathcal{R}'\left(N_{n_{s-1}}^2 f,\xi\right)^{(1)}\,d\xi\\ &- e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi}\mathcal{R}'\left(N_{n_{s-1}}^2 f,\xi\right)^{(1)}\,d\xi\Bigg).\end{aligned}$$

After computing the integrals and regrouping terms, we obtain

$$\begin{aligned}I(a_t,f) =& e^{\lambda_- t}\Theta_-^{(2)}(f) + e^{\lambda_+ t}\Theta_+^{(2)}(f) - e^{\lambda_- t}(t-1)\Theta_{-,t}^2(f)\\ &+\mathcal{R}\left(f,t\right)^{(2)},\end{aligned}$$

where the coefficients are redefined as

$$\begin{aligned}&\Theta_-^{(2)}(f) = \Theta^-(f) - \sum_{s=2}^{n}\Theta_+^{(0)}(N_{n_{s-1}}^2 f) - \sum_{s=2}^{n}\int_1^\infty e^{(n-3+\lambda_+)\xi}\mathcal{R}'(N_{n_{s-1}}^2 f,\xi)^{(1)}\,d\xi,\\ &\Theta_+^{(2)}(f) = \Theta^+(f), \Theta_{-,t}^2(f) = \sum_{s=2}^{n}\Theta_+^{(0)}(N_{n_{s-1}}^2 f).\end{aligned}$$

The error term after the second iteration becomes

$$\begin{aligned}\mathcal{R}'\left(f,t\right)^{(2)} =& -\left(\frac{2t-1}{4}\right)e^{(\lambda_- -2)t}\sum_{s=2}^{n}\Theta_1(N_{n_{s-1}}^2 f)\\ &+e^{(\lambda_- -2)t}\sum_{s=2}^{n}\Theta_1(N_{n_{s-1}}^2 f) + e^{(\lambda_- -2)t}(t-1)\frac{1}{3}\sum_{s=2}^{n}\Theta_1(N_{n_{s-1}}^2 f)\\ &+e^{(\lambda_- -2)t}\frac{1}{9}\sum_{s=2}^{n}\Theta_1(N_{n_{s-1}}^2 f) + \mathcal{R}\left(f,t\right)^{(2)}.\end{aligned}$$

This term $\mathcal{R}\left(f,t\right)^{(2)}$ can be estimated as

$$\left\|\mathcal{R}\left(f,t\right)^{(2)}\right\| \ll_n e^{-\frac{1}{2}(n+3)t}\left\|f\right\|_{W_{2k+7}}.$$

This estimate on the error term $\mathcal{R}'(N^2_{n_{s-1}}f,t)$ also ensures that all the coefficients Θ are bounded.

Now we consider the next case where $\nu \in \left(1, \frac{n-1}{2}\right)$. Adopting the formula **??** for $\nu > 1$, the asymptotic formula after the first iteration becomes

$$
\begin{aligned}
I(a_t,f) =& e^{\lambda_- t}\left(\frac{e^{2\nu-2}}{2\nu-2}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}}f) + \frac{1}{\nu e^2}\sum_{s=2}^{n}\Theta^-(N^2_{n_{s-1}}f) + \Theta^-(f)\right) \\
& + e^{\lambda_+ t}\Theta^+(f) + e^{(\lambda_- -2)t}\frac{1}{2(1+\nu)}\sum_{s=2}^{n}\Theta^-(N^2_{n_{s-1}}f) \\
& - e^{(\lambda_+ -2)t}\frac{1}{2\nu-2}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}}f) + \mathcal{R}(f,t)^{(1)}.
\end{aligned}
$$

After regrouping the terms, it becomes

$$
I(a_t,f) = e^{\lambda_+ t}\Theta^+(f) - e^{(\lambda_+ -2)t}\frac{1}{2\nu-2}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}}f) + \mathcal{R}'(f,t)^{(1)},
$$

where

$$
\begin{aligned}
\mathcal{R}'(f,t)^{(1)} =& e^{\lambda_- t}\left(\frac{e^{2\nu-2}}{2\nu-2}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}}f) + \frac{1}{\nu e^2}\sum_{s=2}^{n}\Theta^-(N^2_{n_{s-1}}f) + \Theta^-(f)\right) \\
& + e^{(\lambda_- -2)t}\frac{1}{2(1+\nu)}\sum_{s=2}^{n}\Theta^-(N^2_{n_{s-1}}f) + \mathcal{R}(f,t)^{(1)}.
\end{aligned}
$$

We estimate the error term as follows:

$$
\begin{aligned}
\left\|\mathcal{R}'(f,t)^{(1)}\right\| \ll_n & \frac{1}{\nu}\sum_{s=2}^{n}\left(e^{\lambda_- t}\int_1^t e^{(n-3+\lambda_+)\xi}\left\|\mathcal{R}(N^2_{n_{s-1}}f,\xi)\right\| d\xi\right. \\
& \left. + e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi}\left\|\mathcal{R}(N^2_{n_{s-1}}f,\xi)\right\| d\xi\right) \\
\ll_\nu & \|f\|_{W_{2k+5}}\left(e^{\lambda_- t}\int_1^t e^{(2\nu-4)\xi}\, d\xi + e^{\lambda_+ t}\int_t^\infty e^{-4\xi}\, d\xi\right)
\end{aligned}
$$

Then it satisfies

$$
\left\|\mathcal{R}'(f,t)^{(1)}\right\| \ll_\nu \begin{cases} e^{(\lambda_+ -4)t}\|f\|_{W_{2k+5}}, & \text{if } \nu \neq 0, \\ e^{\lambda_- t}t\,\|f\|_{W_{2k+5}}, & \text{if } \nu = 2, \end{cases}
$$

In order to shorten we denote

$$
\Theta^{(1)}_{+,-2}(f) = \frac{1}{2\nu-2}\sum_{s=2}^{n}\Theta^+(N^2_{n_{s-1}}f)
$$

Therefore, the final result for $\nu \in (1,2)$ is shown as

$$
I(a_t,f) = e^{\lambda_+ t}\Theta^+(f) - e^{(\lambda_+ -2)t}\Theta^{(1)}_{+,-2}(f) + \mathcal{R}'(f,t)^{(1)}
$$

where the coefficient $\Theta^+(f)$ and $\Theta^{(1)}_{+,-2}(f)$ satisfies

$$\left\|\Theta^+(f)\right\| \ll_\nu \|f\|_{W_{2k+3}}, \left\|\Theta^{(1)}_{+,-2}(f)\right\| \ll_\nu \|f\|_{W_{2k+5}}.$$

and the error term $\left\|\mathcal{R}'(f,t)^{(1)}\right\| \ll_\nu e^{(-\frac{1}{2}(n+3)+\nu-2)t}\|f\|_{W_{2k+5}}$.
We now study the case $\nu = 2$. We obtain

$$\begin{aligned} I(a_t, f) &= e^{\lambda_+ t}\Theta^+(f) + e^{\lambda_- t}\Theta^-(f) \\ &\quad - \frac{1}{2}\sum_{s=2}^{n}\Big[e^{\lambda_- t}\int_1^t e^{(n-3+\lambda_+)\xi}\big(e^{\lambda_+\xi}\Theta^+(N^2_{n_{s-1}}f) + e^{(\lambda_+-2)\xi}\Theta^{(1)}_{+,-2}(N^2_{n_{s-1}}f)\big)\,d\xi \\ &\quad + e^{\lambda_+ t}\int_t^\infty e^{(n-3+\lambda_-)\xi}\big(e^{\lambda_+\xi}\Theta^+(N^2_{n_{s-1}}f) + e^{(\lambda_+-2)\xi}\Theta^{(1)}_{+,-2}(N^2_{n_{s-1}}f)\big)\,d\xi\Big] \\ &\quad + \mathcal{R}(f,t)^{(2)}. \end{aligned}$$

After iterating $l = \lfloor \nu/2 \rfloor + 1$ times, the asymptotic formula becomes

$$I(a_t, f) = \sum_{i=0}^{l} e^{(\lambda_+-2i)t}\Theta^{(l)}_{+,-2i}(f) + \mathcal{R}'(f,t)^{(l)},$$

Since $\nu > 1$, for any $i = 0, \ldots, l$, the vector $\Theta^{(i)}$ satisfies $\left\|\Theta^{(i)}\right\| \ll_\nu \|f\|_{W_{2k+3+2i}}$. and the error term satisfies

$$\left\|\mathcal{R}'(f,t)^{(l)}\right\| \ll_\nu e^{(\lambda_+-2l)t}\|f\|_{W_{2k+3+2l}}.$$

6 Asymptotics of general functions

In this section, we examine an general function $f : G \to H$ where (π, H, G) is a unitary spherical representation. The function $f : G \to H$ can be decomposed as $f_\nu : G \to H_\nu$, which is studied in the theorem 5.1. Furthermore, as the decomposition of irreducible representation H_ν to the K-component $H_{\nu,\varpi}$, it can be decomposed into $f_{\nu,\varpi}$, such that $f = \sum_\nu \sum_\varpi f_{\nu,\varpi}$. We may also use the notation $f_{\mu,\varpi} \in H_{\mu,\varpi}$ for the decomposition, depending on which parameter is more convenient to use.
And we recall the notation, for a cocompact lattice Γ, such that $H = L^2(\Gamma\backslash G)$, the representation π only has a discrete spectrum. We denote by Σ_p the discrete subset of Spec_p and by Σ_f the finite subset of the complementary series and the trivial representation.

Lemma 8. *Let k_1 to be natural numbers, and let $\mu \in \Sigma_p$ with μ defined by the relationship $\mu = \nu^2 - \frac{(n-1)^2}{4}$, where $\nu \in i\mathbb{R}_{\geq 0}$. Then define the infinite series*

$$C_{k_1}(\mu, \varpi)^{(1)} = \sum_{\mu\in\Sigma_p}\sum_{\varpi}\frac{1}{(1+\mu-2\varpi)^{k_1}}, \tag{27}$$

These series are summable if and only if the condition $k_1 > \frac{n}{2}+1$ is fulfilled.

Proof. For the principal series, the spectral parameter is given by

$$\mu = \nu^2 - \frac{(n-1)^2}{4}, \quad \nu \in i\mathbb{R}_{\geq 0}.$$

It is convenient to rewrite the spectral parameter as

$$\mu = -b_l^2 - \frac{(n-1)^2}{4},$$

and denote the increasing sequence by $b_1 \leq b_2 \leq \cdots \leq b_l \leq \cdots$, where $ib_l \in \Sigma_p \subset i\mathbb{R}_{\geq 0}$. By Weyl's law 3.3, the multiplicities of the b_l satisfy an asymptotic equality 3.3. And we know the property of the eigenvalue of K_{Ω_K} in 6. Our goal is to study the convergence of the series which is equivalent to the convergence $\sum_l \frac{1}{b_l^{2k}}$, for some $k > 0$. Therefore, the sum can be estimated by integrating over the density of terms by the bound

$$\begin{aligned}
\sum_{l,|b_l|>t} \sum_{\varpi} \frac{1}{(b_l^2 + \varpi)^{k_1}} &\approx \int_t^\infty \int_0^\infty \frac{1}{(x^2+y)^{k_1}} \, dy \, dN(x) \\
&\ll_n \int_t^\infty \int_0^\infty \frac{x^{n-1}}{(x^2+y)^{k_1}} \, dy \, dx, \\
&\ll_n \int_t^\infty x^{n+1-2k_1} \, dx
\end{aligned}$$

In order to keep the integral to converge, we need the exponent of x to be less than -1, i.e. $k_1 > \frac{n}{2} + 1$. □

The relationship between the Sobolev norm of the general function f and the Sobolev norm of the joint eigenfunction $f_{\mu,\varpi}$ is established in the following proposition.

Proposition 3. *For $k \geq 2$ and for any Casimir eigenvalue $\mu \in \Sigma_p \cup \Sigma_f$ associated to the parametere ν of the representation (π, H, G) and the Casimir eigenvalue ϖ of the representation (ω, K), there exist positive constants $C_k(\mu, \varpi)$, which depend solely on k and the Casimir eigenvalues μ and ϖ, such that the following holds.*
Consider $s \in \mathbb{R}_{\geq 0}$ and a function f belonging to the Sobolev space $W_{s+k}(H)$ of order $s+k$, as defined in 1. Denote by $f_{\mu,\varpi}$ the orthogonal projection of f onto the closed subspace $W_{s+k}(H_{\mu,\varpi})$, respecting the inner product on the Sobolev space $W_{s+k}(H)$. Then, we have

$$\sum_{\mu \in \Sigma_p \cup \Sigma_f} \sum_{\varpi} \|f_{\mu,\varpi}\|_{W_{s+k}} \leq C_k(\mu, \varpi) \|f\|_{W_{s+k}}.$$

Proof. The proof of this proposition is a simple consequence of the Cauchy-Schwarz inequality. Recall from the lemma that for any μ, ϖ, the Sobolev norm satisfies

$$(1 + \mu - 2\varpi)^{-k} \|f_\mu\|^2_{W_{s+k}} = \|f_\mu\|^2_{W_s}.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\sum_{\mu\in\Sigma_p\cup\Sigma_f}\sum_{\varpi}\|f_{\mu,\varpi}\|_{W_s} &= \sum_{\mu\in\Sigma_p\cup\Sigma_f}\sum_{\varpi}(1+\mu-2\varpi)^{-k/2}\|f_\mu\|_{W_{s+k}} \\ &\le \left(\sum_{\mu\in\Sigma_p\cup\Sigma_f}\sum_{\varpi}(1+\mu-2\varpi)^{-k}\right)^{1/2}\left(\sum_{\mu\in\Sigma_p\cup\Sigma_f}\sum_{\varpi}\|f_\mu\|^2_{W_{s+k}}\right)^{1/2}.\end{aligned}$$

We define the constant

$$C_k\,(\mu,\varpi)^{(1)} = \left(\sum_{\mu\in\Sigma_p\cup\Sigma_f}\sum_{\varpi}(1+\mu-2\varpi)^{-k}\right)^{1/2},$$

which is finite due to the lemma 8 above. Therefore, by Parseval's identity,

$$\|f\|^2_{W_{s+k}} = \sum_{\mu\in\Sigma_p\cup\Sigma_f}\sum_{\varpi}\|f_\mu\|^2_{W_{s+k}},$$

the inequality is proved. □

Consider a unitary spherical representation (π, H, G). We now have all the necessary ingredients to prove the asymptotic formula for the general function $f : G \to H$. The function f can be decomposed into components $f_{\nu,\varpi} : G \to H_{\nu,\varpi}$ such that

$$\begin{aligned}f = &\sum_{\nu\in i\mathbb{R}_{>0}}\sum_{\varpi} f_{\mu,\varpi} + f_0 + \sum_{\nu\in(0,1)}\sum_{\varpi} f_{\mu,\varpi} + f_1 \\ &+ \sum_{\nu\in(1,2)}\sum_{\varpi} f_{\nu,\varpi} + f_2 + \sum_{\nu\in\left(2,\frac{n-1}{2}\right)}\sum_{\varpi} f_{\mu,\varpi}\end{aligned}$$

where the function f_0, f_1 and f_2 are given by

$$f_0 = \sum_{\varpi} f_{\nu=0,\varpi},\,,\, f_1 = \sum_{\varpi} f_{\nu=1,\varpi},\, f_2 = \sum_{\varpi} f_{\nu=2,\varpi}.$$

And δ_0 and δ_i with $i = 1, 2$ is defined as follows:

$$\delta_0 = \begin{cases}1, & \text{if } \mu = -\frac{(n-1)^2}{4} \in \Sigma_p, \\ 0, & \text{if } \mu = -\frac{(n-1)^2}{4} \notin \Sigma_p.\end{cases}, \delta_i = \begin{cases}1, & \text{if } \mu = i - \frac{(n-1)^2}{4} \in \Sigma_f, \\ 0, & \text{if } \mu = i - \frac{(n-1)^2}{4} \notin \Sigma_f.\end{cases},$$

Thus we apply the theorem 5.1, which delivers on a formal level the equality. Then the spherical average $I(a_t, f)$ is written as

$$\int_K f(a_t k)\,dk = \sum_{\mu\in\Sigma_p}\sum_{\varpi}\int_K f_{\mu,\varpi}(a_t k)\,dk + \sum_{\mu\in\Sigma_f}\sum_{\varpi}\int_K f_{\mu,\varpi}(a_t k)\,dk$$

Plugging the asymptotic formulae corresponding to each irreducible representation into above expression, we obtain the final asymptotic formula of the spherical average $I(a_t, f)$ for an arbitrary function f, Then the spherical average $I(a_t, f)$ is written as

$$\int_K f(a_t k)\, dk = \int_K f(k)\, dk + I(a_{t,f_p}) + I\left(a_t, f_c\right) + \mathcal{R}_{all}(f,t).$$

where the asymptotic formula for the principal series $I(a_t, f_p)$ is given by

$$\begin{aligned} I(a_t, f_p) = &\sum_{\nu \in i\mathbb{R}_{>0}} \sum_{\varpi} \left(e^{\lambda_- t}\Theta^-(f_{\nu,\varpi}) + e^{\lambda_+ t}\Theta^+(f_{\nu,\varpi})\right) \\ &+ \delta_0 \Bigg(e^{\frac{1}{2}(1-n)t}\Theta^-(f) + te^{\frac{1}{2}(1-n)t}\Theta^+(f) \\ &+ e^{-\frac{1}{2}(3+n)t}(1+4t)\Theta^{(1)}_t \\ &+ e^{-\frac{1}{2}(3+n)t}(4t^2+3t+1)\Theta^{(1)}_{t^2} \Bigg). \end{aligned}$$

The asymptotic formula for the complementary series $I(a_t, f_c)$ is given by

$$\begin{aligned} I(a_t, f_c) = &\sum_{\nu \in (0,1)} \sum_{\varpi} \left(e^{\lambda_- t}\Theta^-(f) + e^{\lambda_+ t}\Theta^+(f) + e^{(\lambda_- - 2)t}\Theta^{(1)}_{-,-2}(f) + e^{(\lambda_+ - 2)t}\Theta^{(1)}_{+,-2}(f) \right) \\ &+ \sum_{\nu \in (1,2)} \sum_{\varpi} \left(e^{\lambda_+ t}\Theta^+(f) - e^{(\lambda_+ - 2)t}\Theta^{(1)}_{+,-2}(f) \right) \\ &+ \sum_{\nu \in \left(2, \frac{n-1}{2}\right)} \sum_{\varpi} \sum_{i=0}^{l} e^{(\lambda_+ - 2i)t}\Theta^{(l)}_{+,-2i}(f) \\ &+ \delta_1 \left(e^{\lambda_- t}\Theta^{(2)}_-(f) + e^{\lambda_+ t}\Theta^{(2)}_+(f) - e^{\lambda_- t}(t-1)\Theta^{(2)}_{-,t} \right) \\ &+ \delta_2 \left(\sum_{i=0}^{2} e^{(\lambda_+ - 2i)t}\Theta^{(2)}_{+,-2i}(f) + e^{\lambda_- t}\Theta^{(2)}_-(f) + e^{\lambda_- t}(t-1)\Theta^{(2)}_{-,t} \right). \end{aligned}$$

for every $t \geq 1$, the residue term $\mathcal{R}_{all}(f,t)$ is defined as

$$\mathcal{R}_{all}(f,t) = \sum_{\nu \in \Sigma_p \setminus \{0\}} \sum_{\varpi} \mathcal{R}_p(f_{\nu,\varpi}, t) + \delta_0 \mathcal{R}_p(f_0, t) + \sum_{\nu \in \Sigma_f} \sum_{\varpi} \mathcal{R}_c(f_{\nu,\varpi}, t).$$

Here, $\mathcal{R}_p(f,t)$ represents the residue term for the irreducible principal series representation, while $\mathcal{R}_c(f,t)$ represents the residual term for the irreducible complementary series representation. The different error terms corresponding to the various cases of irreducible representations are calculated in detail in the theorem 5.1.

Consider the norm of the residual term $\mathcal{R}_{all}(f,t)$. It can be bounded by the sum of three distinct terms, as shown below:

$$\|\mathcal{R}_{all}(f,t)\| \leq \sum_{\nu \in \Sigma_p \setminus \{0\}} \sum_{\varpi} \|\mathcal{R}_p(f_{\nu,\varpi}, t)\| + \delta_0 \|\mathcal{R}_p(f_0, t)\| + \sum_{\nu \in \Sigma_f} \sum_{\varpi} \|\mathcal{R}_c(f_{\nu,\varpi}, t)\|.$$

The last two terms, for $\nu = 0$ and $\nu \in \Sigma_f$, involve only a finite sum. Using the proposition and the corresponding formulae in the theorem, we obtain:

$$\begin{aligned}
\delta_0 \|\mathcal{R}_p(f_0,t)\| + \sum_{\nu\in\Sigma_f} \|\mathcal{R}_c(f_\nu,t)\| \ll_\nu \delta_0 t^3 e^{-\frac{1}{2}(n+7)t} \|f_\mu\|_{W_{2k+5}} &+ \sum_{\nu\in(0,1)} e^{\left(-\frac{1}{2}(7+n)+\nu\right)t} \|f_\mu\|_{W_{2k+3}} \\
+ \delta_1 e^{-\frac{1}{2}(3+n)t} \|f_1\|_{W_{2k+7}} &+ \sum_{\nu\in(1,2)} e^{\left(-\frac{1}{2}(3+n)+\nu-2\right)t} \|f_\mu\|_{W_{2k+5}} \\
+ \delta_2 e^{-\frac{1}{2}(3+n)t} \|f\|_{W_{2k+7}} &+ \sum_{\nu>2} e^{(\lambda_+ -2l)t} \|f\|_{W_{2k+3+2l}} \\
\ll_\nu e^{-\frac{1}{2}(3+n)t} &\sum_{\nu\in\Sigma_f\cup\{0\}} \|f\|_{W_{2k+3+2l}} .
\end{aligned}$$

The main contribution comes from the infinite summation over $\nu \in \Sigma_p \backslash \{0\}$:

$$\begin{aligned}
\sum_{\nu\in\Sigma_p\backslash\{0\}} \sum_{\varpi} \|\mathcal{R}_p(f_{\nu,\varpi},t)\| &\ll_n e^{-\frac{1}{2}(n+3)t} \sum_{\nu\in\Sigma_p\backslash\{0\}} \sum_{\varpi} \max\left(1, \frac{1}{|\nu|^2}\right) \|f_{\nu,\varpi}\|_{W_{2k+3}} \\
&\ll_n e^{-\frac{1}{2}(3+n)t} \left(\sum_{|\nu|\in(0,1)} \frac{1}{|\nu|^2} \sum_{\varpi} \|f_{\nu,\varpi}\| + \sum_{|\nu|\geq 1} \|f_{\nu,\varpi}\|_{W_{2k+3}} \right) .
\end{aligned}$$

Since we consider $\nu \in \Sigma_p$ for the principal series spectrum, which is a discrete subset of $i\mathbb{R}_{>0}$ and has only finite contribution within the fixed interval $(0,1)$ due to Weyl's law 3.3, we obtain

$$\sum_{\nu\in\Sigma_p\backslash\{0\}} \sum_{\varpi} \|\mathcal{R}_p(f_{\nu,\varpi},t)\| \ll_\nu e^{-\frac{1}{2}(n+3)t} \sum_{\nu\in\Sigma_p\backslash\{0\}} \sum_{\varpi} \|f_{\nu,\varpi}\| .$$

Thus, by applying the previous proposition, the error term $\mathcal{R}_{all}(f,t)$ satisfies

$$\begin{aligned}
\|\mathcal{R}_{all}(f,t)\| &\ll_\nu e^{-\frac{1}{2}(n+3)t} \sum_{\nu\in\Sigma_p\backslash\{0\}} \sum_{\varpi} \|f_{\nu,\varpi}\| + e^{-\frac{1}{2}(3+n)t} \sum_{\nu\in\Sigma_f\cup\{0\}} \|f\|_{W_{2k+3+2l}} \\
&\ll_\nu e^{-\frac{1}{2}(3+n)t} \|f\|_{W_{2k+2l+3}} .
\end{aligned}$$

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