Contents

1 Introduction

Throughout this draft, the p-adic group is denoted as $G_p = SL_2(\mathbb{Q}_p)$, in which $\mathbb{Q}_p$ is the p-adic completion of the rational number $\mathbb{Q}$. Denote the p-adic integers as $\mathbb{Z}_p$, where all units are as $\mathbb{Z}_p^\times$. $\mathbb{R}$ is identified as $\mathbb{Q}_\infty$ and denotes $G_\infty = SL_2(\mathbb{R})$. Let Γ be a lattice of $G = G_\infty \times G_p$. We want to understand how the orbits of the analogue of horocycle flow distribute over the homogeneous space $X = \Gamma \backslash (G_\infty \times G_p)$, on which $G_\infty \times G_p$ acts by left translations, respectively, i.e. for a point $z \in X$, z can be written as $\Gamma(g_\infty^z, g_p^z)$, where (g_∞^z, g_p^z) is the representative of z. For any $(g_\infty, g_p) \in G$, the action is defined as

$$\Gamma(g_\infty^z, g_p^z) \cdot (g_\infty, g_p) = \Gamma(g_\infty^z g_\infty, g_p^z g_p).$$

Given each point z_∞ in the real homogeneous space $\Gamma' \backslash G_\infty$ where Γ' is a lattice of G_∞, the non-closed orbits starting from point z_∞ become asymptomatically equidistributed on $\Gamma' \backslash G_\infty$, as proven by Dani and Smillie **??**. Later, around the early nineties, M. Ratner completed the uniform distribution for unipotent flow in a series of papers over real or S-adic Lie groups **???**. For this draft, we obtain the effective version of M. Ratner's results in the S-adic homogeneous space. In the interest of the effective version of the equidistribution problem, over the real group G_∞ with a cocompact lattice Γ', such estimates were obtained by Burger **?**. Furthermore, in the more general space $\Gamma' \backslash G_\infty$ that is non-compact but of finite volume, due to the existence of closed horocycle orbits, which are dense in the homogeneous space $\Gamma' \backslash G_\infty$, the rate of convergence of non-closed horocycle orbits is sensitively dependent on the starting point Γg_∞,which is known from Strömbergsson **?**. However, thanks to the totally different topological property of the p-adic space G_p/K_p, the p-adic analogue of the upper half complex plane G_∞/K_∞ is called the Bruhat-Tits tree. It has a very different nature from the hyperbolic upper half plane, more details in Casselman **?**.

The main idea to estimate the equidistribution of the horocycle flow over p-adic homogeneous space is to study the decay rate of matrix coefficient of unitary representation of G_p, which draws attention not only in homogeneous dynamics but also in many other fields. The representation tool that we used in the proof is collected in section 2 from the book **?**.

About the effective bound of the convergence rate of equidistribution of the non-closed orbits of horocycle flow in the space $X = \Gamma \backslash (G_\infty \times G_p)$, here is the basic statement about this problem: Let $\sigma : G \to X$ be the natural projection defined by $p(g) = \Gamma g$. For any fixed point $z \in X$, the projection $\sigma_z : G \to X$ is defined as $p_z(g) = zg$. On G_∞ or G_p, we have the usual Haar measures μ_∞ or μ_p induced from the corresponding invariant

differential form. For the product topological group $G = G_\infty \times G_p$, the measure is defined by product: $\mu = \mu_\infty \otimes \mu_p$. Also denote by $\overline{\mu}$ the $G-$ invariant probability measure in X induced by μ, which is defined in the standard way.
Consider the subgroups of G_p, we define the maximal compact subgroup $SL_2(\mathbb{Z}_p)$ as K_p and

$$
\begin{aligned}
A_p &= \left\{ a_p(\omega, n) = \begin{pmatrix} \omega p^n & 0 \\ 0 & \omega^{-1}p^{-n} \end{pmatrix} \mid n \in \mathbb{Z}_{\geq 0} \text{ and } \ \omega \in \mathbb{Z}_p^\times \right\}, \\
H_p &= \left\{ h_p(t_p) = \begin{pmatrix} 1 & t_p \\ 0 & 1 \end{pmatrix} \mid t_p \in \mathbb{Q}_p \right\}, \\
H_p^- &= \left\{ h_p^-(t_p^-) = \begin{pmatrix} 1 & 0 \\ t_p^- & 1 \end{pmatrix} |t_p^- \in \mathbb{Q}_p \right\}.
\end{aligned}
$$

For the elements in A_p, we shorten the notation as $a_n = a_p(1, n)$. And we omit the variables to shorten the notation to a_p, h_p, h_p^- if they are not needed in the context. Similarly, subgroups of G_∞ can be defined as follows: The maximal compact subgroup $SO(2, \mathbb{R})$ is denoted as K_∞,

$$
\begin{aligned}
K_\infty &= \{r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R}\}, \\
A_\infty &= \left\{ a_\infty(x_\infty) = \begin{pmatrix} e^{x_\infty/2} & 0 \\ 0 & e^{-x_\infty/2} \end{pmatrix} \mid x_\infty \in \mathbb{R}_{\geq 0} \right\}, \\
H_\infty &= \left\{ h_\infty(t_\infty) = \begin{pmatrix} 1 & t_\infty \\ 0 & 1 \end{pmatrix} \mid t_\infty \in \mathbb{R} \right\}, \\
H_\infty^- &= \left\{ h_\infty^-(t_\infty^-) = \begin{pmatrix} 1 & 0 \\ t_\infty^- & 1 \end{pmatrix} \mid t_\infty^- \in \mathbb{R} \right\}.
\end{aligned}
\tag{1}
$$

Fix Haar measures ν_∞ on H_∞ and ν_p on H_p, and ν on $H = H_\infty \times H_p$, given a function f defined on the homogeneous space X, for $z = \Gamma(g_\infty, g_p) \in X$ that are not in any periodic $H_\infty \times H_p$ orbits, as the volume of $B = B_\infty \times B_p$ growing, that is defined as follow:

$$
\begin{aligned}
B_\infty(R_\infty) &= \{h_\infty(t_\infty) \in H_\infty \mid |t_\infty|_\infty \leq R_\infty\}, \\
B_p(R_p) &= \left\{h(t_p) \in H_p \mid |t_p|_p \leq R_p\right\}.
\end{aligned}
$$

We want to study ergodic average,

$$
A_B(f)(z) = \frac{1}{\nu(B)} \int_B f((h_\infty, h_p) \cdot z)\, d\nu.
$$

The goal is to obtain the effective bound of the convergence rate of this average. In the context of the product space X, a distinct methodology is required for the p-adic

component compared to an isolated homogeneous space in the real case. In our first proof, the idea is a continuation of Margulis-Kleinbock thickening method **?**. The benefits coming from p-adic place is that the smooth functions give us local constancy, which means that our result is optimal in the sense of p-adic component. Here we used the Kunze-Stein inequality, which was first studied by Cowling **?** and generalized to homogeneous tree by Alessandro Veca **?**. Then combining with the well-known results for the real component by Burger and Strömbergsson **??**, the quantative equidistribution of non-closed horocycle orbits is shown in any non-compact but of finite volume homogeneous space X.
In order to avoid the loss from the thickening process, the second proof using representation by Burger is given.
Is there more history about padic work. Then explain the structure of this paper.
Other related work?
structure of this paper
In this paper, we want to study the quantitative estimate of the equidisbution of non-closed horocycle orbits on S-adic homogeneous space where S is a finite subset of all possible places. The first part is to estimate the distribution of non-closed orbits of horocycle flow on a p-adic homogeneous space by Margulis's thickening method **?**. In the second part with more abstract representation theory, we can study the quantative equidistribution of horocycle flow in the homogeneous space over $\mathbb{R} \times \mathbb{Q}_p$.
Should I add more about the basic knowledge about p-adic numbers

2 Representation theory of G

In this section, let us review the decay rate of the matrix coefficient of the unitary representation of G_p and the representation theory of the product group $G_\infty \times G_p$.

Definition 1 (Tempered representation)**.** Given a unitary representation (π, H_π, G), it is called a tempered representation if there exists $\mathcal{V}$ dense in H_π, for any $v \in \mathcal{V}$ and $\epsilon > 0$,

$$< \pi(g)v, v > \in L^{2+\epsilon}(G).$$

There is a rough classification of the irreducible unitary representations of the p-adic group G_p with respect to the decay property of matrix coefficients.

Theorem 2.1 (Classification of representations of G_p **?**)**.** *The unitary irreducible representations of G_p, denoted by (π, H), fall into the following three categories: the trivial* *representation* 1_G*, the tempered representations* (π, H_π)*, and the complementary series representations* (γ^s, H_s) *for* $s \in (0,1)$*, i.e.*

$$\widehat{G_p} = \{1_G\} \cup \{\textit{temerped } \pi \in \widehat{G_p}\} \cup \{\gamma^s \mid s \in (0,1)\}.$$

For the tempered representation defined above 1, its matrix coefficient has a better decay rate. For such a class of representations of the real or p-adic group, the matrix coefficient is bounded by the Harish-Chandra function in **?**.

Theorem 2.2 (Decay rate of matrix coefficient of representation of $G_p = SL_2(\mathbb{Q}_p)$)**.** *For a irreducible unitary representation* $(\pi, L^2(G_p))$ *of* G_p*, then for any* K_p*-finite vector* ψ, ϕ *in* $L^2(G_p)$*, such that*

$$< \pi(a_n)\phi, \psi > \leq \sqrt{d_K(\phi) d_K(\psi)} ||\phi||_2 ||\psi||_2 \varpi_\pi(a_n).$$

in which $a_n \in A_p$ *and* ϖ_π *is denoted as the spherical function corresponding to the representation* π*.*

And the explicit expression of the p-adic Harish-Chandra function is shown in theorem 2.3. With the classification of the representations of G_p, there is an explicit asymptotic formula for the spherical function ϖ in each case.

Theorem 2.3 (Asymptotic formulae of spherical functions **?**)**.** *If the representation* (π, H_π) *of* G_p *is tempered, then the Harish-Chandra function is*

$$\Xi(a_n) = \frac{p^{-n}}{p+1}((2n+1)(p-1)+2).$$

Similarly, for the complementary series representation (γ^s, H_s) *of* G_p*, the spherical function has the following bound,*

$$\varpi(a_n) \ll \frac{p^{n(s-1)}}{1-p^{-s}}.$$

For the unitary representation of the group $G_\infty \times G_p$, we need to introduce some basic concepts and definitions. The study of unitary representation of G_∞ is dependent on Bargmann's classification; more details can be found in **?**. And the expression of a p-adic representation is symmetric to the real part.

To study the general unitary representation, here is a short review about direct integral decomposition of unitary representation. The discussion begins with the unitary representation of G_∞. Let $(Z_\infty, \mu_{Z_\infty})$ be a Borel measure space, for each $\xi_\infty \in Z_\infty$, $(\pi_{\infty,\xi_\infty}, \mathcal{H}_{\infty,\xi_\infty})$ is an irreducible unitary representation of G_∞. The direct integral

$$\pi_\infty = \int_{Z_\infty}^{\oplus} \pi_{\infty,\xi_\infty} \, d\mu_{Z_\infty}(\xi_\infty), \mathcal{H}_\infty = \int_{Z_\infty}^{\oplus} \mathcal{H}_{\infty,\xi_\infty} \, d\mu_{Z_\infty}(\xi_\infty)$$

is defined as a unitary representation $(\pi_\infty, \mathcal{H}_\infty)$ of G_∞. Fixed the basis

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

of the corresponding Lie algebra $\mathfrak{g}_\infty$ of G_∞. And the Casimir operator is defined for the C^2-vectors in the Hilbert space $\mathcal{H}_\infty$ by the formula

$$\Omega_\infty = -\frac{1}{4}(Y^2 + 2X_+X_- + 2X_-X_+).$$

And let $\Delta = -\sum\limits_i X_i^2 \in U(\mathfrak{g}_\infty)$ where $\{X_i\}$ is any fixed basis of $\mathfrak{g}_\infty$. Then the closure $\overline{\pi_\infty(\Delta)}$ of $\pi_\infty(\Delta)$ is a self-adjoint operator on $\mathcal{H}_\infty$ **?**.

Given an irreducible unitary representation $(\pi_{\infty,\xi_\infty}, \mathcal{H}_{\infty,\xi_\infty})$ of G_∞ where $\mathcal{H}_\infty$ is a separable complex Hilbert space with basis

$$\Phi_\infty(\xi_\infty) = \{\phi^{(n)}_{\infty,\xi_\infty} = \phi^{(n)}_\infty(\xi_\infty) \mid n \in \Sigma \subseteq 2\mathbb{Z}\}.$$

In this case that π_{∞,ξ_∞} is irreducible, the Casimir operator $\pi_{\infty,\xi_\infty}(\Omega_\infty) = \lambda id$ acts as a scalar on the space of smooth vectors $\mathcal{H}^\infty_{\infty,\xi_\infty}$.

According to Bargmann's classification of irreducible unitary representations of G_∞,

1. when $\lambda > 0, \Sigma = 2\mathbb{Z}$, the irreducible representation $(\pi_{\infty,\xi_\infty})$ with ξ_∞ is a principal series representation or a complementary representation.

2. when $\lambda = 0, \Sigma = \{0\}$, it is just a trivial representation.

3. when $\lambda = \frac{m}{2}\left(1 - \frac{m}{2}\right)$, $\Sigma = \{m, m+2, m+4, \dots\}$ or $\{-m, -m-2, -m-4, \dots\}$, it is a discrete series representation.

Now adopting the information of irreducible representations in decomposition, for almost all $\xi \in Z$, $\lambda = \lambda(\xi)$, the Casimir operator

$$\pi_{\infty,\xi}(\Omega_\infty) = \lambda(\xi) id$$

in $H_{\infty,\xi}$ which has a basis $\{\phi^{(n)}_{\infty,\xi} = \phi^{(n)}_\infty(\xi) \mid n \in \mathbb{Z}\}$. And these bases can be chosen as $K_\infty-$eigenvectors, i.e.

$$\pi_{\infty,\xi}(r(\theta))\phi^{(n)}_{\infty,\xi} = e^{in\theta}\phi^{(n)}_{\infty,\xi}.$$

The eigenvalue λ can be parameterized as follows, for a unique number $s = s(\xi_\infty)$ such that $\lambda = s(1-s)$ and $\mathrm{Re}(s) \geq \frac{1}{2}, \mathrm{Im}(s) \geq 0$. Then $s \in \frac{1}{2} + i\mathbb{R}$ if $\lambda \geq \frac{1}{4}$, $s \in (\frac{1}{2}, 1]$ if $0 \leq \lambda < \frac{1}{4}$, $s \in \mathbb{Z}^+$ if $\lambda < 0$.

1. $Z^0_\infty \cup Z^{rs}_\infty$ is discrete measure spaces, for trivial $\pi_{\infty,\xi} = 1$, $\mu_{Z_\infty}(\{\xi_\infty\}) = 1$, for any $\xi_\infty \in Z^0_\infty \cup Z^{rs}_\infty$,

2. Z^{rs}_∞ is finite. And there exists unique $\xi \in Z^{rs}$, such that $\pi_{\infty,\xi}$ is trivial. For all the other $\xi \in Z^{rs}$, we have $s(\xi) \in (\frac{1}{2}, 1)$,

3. For any $\xi \in Z^{ct}_\infty$, $\pi_{\infty,\xi}$ is principal series representation and $s(\xi) \in \frac{1}{2} + i\mathbb{R}_{\geq 0}$.

Then given a Borel measure (Z, μ), the unitary representation $(\pi, \mathcal{H})$ of the product group $G_\infty \times G_p$ can be written as direct integral

$$\pi = \int_Z^{\oplus} \pi_{\infty,\xi} \otimes \pi_{p,\xi} \, d\mu_Z(\xi),$$

and the representation space is the Hilbert space $\mathcal{H}$ with direct integral decomposition

$$\int_Z^{\oplus} \mathcal{H}_{\infty,\xi} \otimes \mathcal{H}_{p,\xi} \, d\mu_Z(\xi).$$

Now, in the direct integral $\mathcal{H} = \int_Z^{\oplus} \mathcal{H}_{\infty,\xi} \otimes H_{p,\xi} \, d\mu(\xi)$, the $\mathcal{H}_\xi = H_{\infty,\xi} \otimes H_{p,\xi}$ can be further classified only with respect to the spectrum decomposition of $H_{\infty,i}$ to the disjoint union of $Z = Z^0 \cup Z^{rs} \cup Z^{ct}$.

For each irreducible representation $\mathcal{H}_{\infty,\xi_\infty} \otimes \mathcal{H}_{p,\xi}$, there are basis defined as

$$\begin{aligned} \Phi_\infty(\xi_\infty) &= \{\phi^{(n)}_{\infty,\xi} = \phi^{(n)}_\infty(\xi) \mid n \in \mathbb{Z}\}, \\ \Phi_p(\xi) &= \{\phi^{(m)}_{p,\xi} = \phi^{(m)}_p(\xi) \mid n \in \mathbb{Z}\}. \end{aligned}$$

And the Hilbert space $\mathcal{H}_\infty \otimes \mathcal{H}_p$ contains all the functions $f_\infty \otimes f_p$ defined on Z. In order to make the definitions clearer, we use those pure tensor product functions to express the formulae. The functions $f_\infty(\xi) \otimes f_p(\xi)$ belong to the irreducible component $\mathcal{H}_{\infty,\xi} \otimes \mathcal{H}_{p,\xi}$ which are $\Phi_\infty \otimes \Phi_p$-measurable, that is, $< f_\infty(\xi), \phi^{(n)}_{\infty,\xi} > \cdot < f_p(\xi), \phi^{(m)}_{p,\xi} >$ is a measurable function of (ξ, ξ) for all $n, m \in \mathbb{Z}$. And all the $\Phi_\infty \otimes \Phi_p$-measurable functions satisfy that

$$\int_Z ||f_\infty(\xi)||^2 \, d\mu_Z(\xi) \cdot \int_Z ||f_p(\xi)||^2 \, d\mu_Z(\xi) < \infty.$$

We also assume that there exists a small eigenvalue $\lambda \in (0, \frac{1}{4})$ in the discrete spectrum, where λ_1 is denoted as the smallest one. And define $s_1 \in (\frac{1}{2}, 1)$ by $\lambda_1 = s_1(1 - s_1)$.

Definition 2 (decay exponent of complementary series)**.** Let $s \in (0, 1)$. Then the complementary series γ^s has an almost decay exponent

$$\kappa_{\gamma^s} = 1 - s$$

where s^2 is the eigenvalue of the Casimir operator.

For a unitary representation $(\pi, \mathcal{H})$ of $G_\infty \times G_p$ with direct integral decomposition $\pi = \int_Z \pi_{\infty,\xi} \otimes \pi_{p,\xi}\, d\mu_Z(\xi)$, there exist decay exponents

$$\kappa_\infty = 1 - s_\infty(\xi), \;\; \kappa_p = 1 - s_p(\xi),$$

for the irreducible family $\pi_{\infty,\xi}$ and $\pi_{p,\xi}$.
The matrix coefficient $< \pi(g)v, w >$ for $g = (g_\infty, g_p)$ and K-finite vectors $v, w \in \mathcal{H}$,

$$\begin{aligned} &< \int_Z \pi_{\infty,\xi} \otimes \pi_{p,\xi}(g)\, d\mu_Z(\xi) \cdot v, w > \\ &= \int_Z < \pi_{\infty,\xi}(g_\infty)v_\xi, w_\xi > \cdot < \pi_{p,\xi}(g_p v_\xi, w_\xi) > \, d\mu_Z(\xi) \end{aligned}$$

For the general unitary representation $(\pi, \mathcal{H}, G_\infty \times G_p)$, the following theorem shows the effective decay rate of its matrix coefficient **?**.

Theorem 2.4. *Given a unitary representation $(\pi, \mathcal{H})$, assume that there exists $\kappa_\infty = min(1 - s(\xi))$ for the complementary series representations in the irreducible family $\pi_{\infty,\xi}$ and $\kappa_p = min(1 - s(\xi))$ same for the complementary series representations in the irreducible family $\pi_{p,\xi}$. Then for any $K_\infty \times K_p$-finite vectors, we have*

$$|< \pi((a_n, a_t))v, w >| \leq \sqrt{d_{K_\infty(v)} d_{k_\infty}(w) d_{K_p}(v) d_{K_p}(w)} ||v||||w|| \Xi_\infty(a_t)^{\kappa_\infty} \Xi_p(g)^{\kappa_p},$$

where Ξ_∞ and Ξ_p are the Harish-Chandra functions.

We also denote by $\|\psi\|_{\mathrm{Lip}}$ the Lipschitz constant of a function ψ on X,

$$\|\psi\|_{\mathrm{Lip}} \stackrel{\mathrm{def}}{=} \sup_{x,y \in X, x \neq y} \frac{|\psi(x) - \psi(y)|}{d_\infty(x, y)},$$

where for two points $x = \Gamma(g_\infty^x, g_p^x)$ and $y = \Gamma(g_\infty^y, g_p^y)$,

$$d_\infty(x, y) = \inf_{\gamma \in \Gamma} \{d_{G_\infty}(g_\infty^x, \gamma g_\infty^y)\}.$$

3 Effective equidistribution of horocycle flow by thickening

We want to understand the effective bound of the convergence rate of $A_B(f)$ in the S-adic homogeneous space X by the thickening method, in which the smooth functions f are defined as follows:

Definition 3. For the smooth functions in group G and quotient X, it is defined as follows.
A function $\phi : G \to \mathbb{R}$ is smooth if for any fixed $g_p \in G_p$, there exists an open subset U of G and a smooth function $\phi_\infty^U : G_\infty \to \mathbb{R}$ such that $\phi((g_\infty, g_p)) = \phi_\infty^U(g_\infty)$.
In the quotient space X, a function $\phi : X \to \mathbb{R}$ is smooth if for any $x \in X$, and for any smooth local section $\sigma^{-1} : U \subset X \to G$ where U is an open subset of X containing x (that is, $\sigma \circ \sigma^{-1} = \mathrm{id}_U$, the identity map on U), the composition $f \circ \sigma : U \to \mathbb{R}$ is a smooth function on U. Here, a local section σ^{-1} selects, for each point in U an element of G that projects to that point via σ.

Observing the subgroups, it is crucial to mention that the inner automorphism Φ of G_p or G_∞ given by $\Phi_n(g) = a_n^{-1} g a_n$ or $\Phi_t(g) = a_t g a_t^{-1}$, is non-expanding in the subgroup of form

$$\widetilde{H} = H^- A = \left\{ \begin{pmatrix} a & 0 \\ t & a^{-1} \end{pmatrix} \right\},$$

for infinite or finite place, i.e. for any radius $r > 0$, consider the ball

$$B_{\widetilde{H}_p}(r) = \{ h_p^- \mid d_{G_p}(h_p^- a_p, \mathrm{id}) \leq r \} \tag{3}$$

in $\widetilde{H}$. The conjugation Φ_n preserves this ball: $\Phi_n(B_{\widetilde{H}_p}(r)) \subseteq B_{\widetilde{H}_p}(r)$.

Lemma 1 (Non-expanding conjugation). *The conjugation $\Phi_n(g) = a_n^{-1} h^- a_p a_n$ is well defined as non-expanding, and the same reasoning for infinite places.*

Proof. For $a_n = \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix}$ and $\begin{pmatrix} a & 0 \\ t_p^- & a^{-1} \end{pmatrix} \in B_{\widetilde{H}_p}$,

$$\begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix} \begin{pmatrix} a & 0 \\ t_p^- & a^{-1} \end{pmatrix} \begin{pmatrix} p^n & 0 \\ 0 & p^{-n} \end{pmatrix} = \begin{pmatrix} a & 0 \\ p^{2n} t_p^- & a^{-1} \end{pmatrix},$$

the p-adic norm of left corner entry $p^{2n} t_p^-$ is shrinking after the conjugation. □

Then by taking expanding conjugation, we can change variables from

$$B_{H_p}(p^{2n}) = \{ h_p \in H_p \mid d_{G_p}(id, h_p) \leq p^{2n} \},$$

to unit ball $B_{H_p}(1)$,

$$\begin{aligned} A_{B_{H_p(p^{2n})}}(f_p, z_p) &= \int_{H_p} \chi_{B_{H_p(p^{2n})}}(h_p(t_p)) f_p(h_p \cdot z_p) \, dt_p \\ &= \int_{H_p} \chi_{B_{H_p(1)}}(h_p) f(a_n^{-1} h_p a_n z_p) \, d\nu_p(h_p). \end{aligned}$$

Then we wish to use the matrix coefficient $< \phi, \pi(g^{-1})f_p >$ to estimate $A_{B_{H_p}}(f_p, z_p)$. To pass from matrix coefficient $< \phi, \pi(g^{-1})f_p >$ to $A_{B_{H_p}}(f_p, z_p)$, we need to choose a suitable function $\phi \in S(X_p)$. By these asymptotic formulae 2.3 of the spherical functions, we can have explicit formulae for the decay rate of matrix coefficients.

Let us choose the normalized Haar measures ν_p^- and ν_p^a in H_p^- and A_p, note that $H_p^- A_p H_p$ is open in $G_p = SL_2(\mathbb{Q}_p)$. so that μ_p is locally the product of ν_p^-, ν_p^a and ν_p, i.e. for $\theta \in S(G_p)$,

$$\int_{H_p^- A_p H_p} \theta(g_p)\, d\mu_p = \int_{H_p^-} \int_{A_p} \int_{H_p} \theta(h_p^- a_p h_p) \Delta(a_p)\, d\nu_p\, d\nu_p^a\, d\nu_p^- . \tag{4}$$

where $\Delta(a_p(\omega, n)) = p^{2n}$. Similarly, let us choose Haar measures $\nu_\infty^-, \nu_\infty^a$ on H_∞^-, A_∞ respectively, normalized so that μ_∞ is locally almost the product of $\nu_\infty^-, \nu_\infty^a$ and ν_∞. By the latter, in view of Lie group **?**, we mean that μ_∞ can be expressed via $\nu_\infty^-, \nu_\infty^a$ and ν_∞ in the following way: for any $\varphi \in L^1(G_\infty)$

$$\begin{aligned} &\int_{H_\infty^- A_\infty H_\infty} \varphi(g_\infty) d\mu_\infty(g_\infty) \\ &= \int_{H_\infty^- \times A_\infty \times H_\infty} \varphi\left(h_\infty^- a_\infty h_\infty\right) \Delta_\infty\left(a_\infty\right) d\nu_\infty^-\left(h_\infty^-\right) d\nu_\infty^a\left(a_\infty\right) d\nu_\infty(h_\infty), \end{aligned} \tag{5}$$

where Δ_∞ is the modular function of (the non-unimodular group) $\widetilde{H_\infty}$.

Here we recall a lemma from **?** to construct the function that we used later in the proof. However, a similar statement of the following lemma in the p-adic setting is trivial.

Lemma 2. *1. For any $r > 0$, there exists a non-negative function $\theta \in C_{comp}^\infty\left(\mathbb{R}^N\right)$ such that* $\operatorname{supp}(\theta)$ *is inside $B(r)$, $\int_{\mathbb{R}^N} \theta = 1$, and $\|\theta\|_\ell \ll r^{-(\ell + N/2)}$.*

2. *Given $\theta_1 \in C_{comp}^\infty\left(\mathbb{R}^N\right)$, $\theta_2 \in C_{comp}^\infty\left(\mathbb{R}^N\right)$, define $\theta \in C_{comp}^\infty\left(\mathbb{R}^N\right)$ by $\theta(x) = \theta_1(x)\theta_2(x)$. Then $\|\theta\|_\ell \ll \|\theta_1\|_\ell \|\theta_2\|_{C^\ell}$.*

3. *Given $\theta_1 \in C_{comp}^\infty\left(\mathbb{R}^{N_1}\right)$, $\theta_2 \in C_{comp}^\infty\left(\mathbb{R}^{N_2}\right)$, define $\theta \in C_{comp}^\infty\left(\mathbb{R}^{N_1+N_2}\right)$ by $\theta\left(x_1, x_2\right) = \theta_1\left(x_1\right)\theta_2\left(x_2\right)$. Then $\|\theta\|_\ell \ll \|\theta_1\|_\ell \|\theta_2\|_\ell$.*

Using Lemma 2, one can choose non-negative functions $\theta_\infty^- \in C_{\text{comp}}^\infty\left(H_\infty^-\right)$, $\theta_\infty^0 \in C_{\text{comp}}^\infty\left(A_\infty\right)$ with

$$\int_{H_\infty^-} \theta_\infty^- = \int_{A_\infty} \theta_\infty^0 = 1$$

such that

$$\operatorname{supp}\left(\theta_\infty^-\right) \cdot \operatorname{supp}\left(\theta_\infty^0\right) \subset B_{\widetilde{H_\infty}}(r_\infty)$$

and at the same time

$$\|\widetilde{\theta_\infty}\|_\ell \ll r_\infty^{(2\ell + N/2)}$$

where $\widetilde{\theta_\infty} \in C_c^\infty(\widetilde{H_\infty})$ is defined by

$$\widetilde{\theta_\infty}\left(h_\infty^- a_\infty\right) \overset{\text{def}}{=} \theta_\infty^-\left(h_\infty^-\right)\theta_\infty^a\left(a_\infty\right)\Delta\left(a_\infty\right)^{-1}.$$

In the p-adic case, for $1 \geq r(a_n z_p) \geq r > 0$ and the fixed point $z_p \in X_p$ such that $\sigma_{a_n z_p}$ is injective in $B_{G_p}(r(a_n z_p))$, we define $r(a_n z_p)$ as the injective radius. By the above lemma 1, we can choose $\theta_p^- \in S(H_p^-)$ and $\theta_p^a \in S(A_p)$ with support $B_{H_p^-}(r_1)$ and $B_{A_p}(r_2)$ such that $B_{H_p^-}(r_1)B_{A_p}(r_2) \subset B_{\widetilde{H_p}}(r(a_n z_p))$ as defined in 3. And in order to normalize the integral, $\theta^- = \frac{\chi_{B_{H_p^-}(r_1)}}{\nu_p^-(B_{H_p^-}(r_1))}$ and $\theta^a = \frac{\chi_{B_{A_p}(r_2)}}{\nu_p^a(B_{A_p}(r_2))}$. The support of functions are explicitly defined as

$$\begin{aligned}
B_{H_p^-}(r_1) &= \{h_p^- \mid d_{G_p}(h_p^-, id) \leq r_1\}, \\
B_{A_p}(r_2) &= \{a_p(\omega, n) \mid d_{G_p}(a_p, id) \leq r_2\}.
\end{aligned}$$

By the property of p-adic metric,

$$d_{G_p}(id, h_p^- a_p) \leq \max\{d_{G_p}(id, a_p), d_{G_p}(a_p, h_p^- a_p)\} = \max\{r_1, r_2\}.$$

So we can just pick $B_{H_p^-}(r_1)B_{A_p}(r_2) = B_{\widetilde{H}_p}(r)$ to simplify the argument. With the lemma, we can choose

$$\phi(h_p^- a_p h_p a_n z_p) = \theta^-(h_p^-)\theta^a(a_p)\chi_{B_{H_p}(1)}(h_p)\Delta(a_p)^{-1}, \tag{6}$$

which is well-defined since the support is contained in the injective radius.

Lemma 3. *The L^2-norm of the function ϕ is bounded by*

$$||\phi||_2 \leq \left(\frac{1}{\nu_p^- \otimes \nu_p^a(B_{\widetilde{H_p}}(r))}\right)^{1/2}$$

Proof. We want to estimate the L^2-norm of ϕ:

$$\begin{aligned}
||\phi||_2^2 &= \int_{X_p} |\phi(x)|^2 \, d\overline{\mu_p} \\
&= \int_{G_p} |\phi(h_p^- a_p h_p a_n z_p)|^2 \, d\mu_p(h_p^- a_p h_p) \\
&= \int_{G_p} \left| \frac{\chi_{B_{H_p^-}(r_1)}(h_p^-)}{\nu_p^-(B_{H_p^-}(r_1))} \frac{\chi_{B_{A_p}(r_2)}(a_p)}{\nu_p^a(B_{A_p}(r_2))} \chi_{B_{H_p}(1)}(h_p)\Delta(a_p)^{-1} \right|^2 \, d\mu_p(h_p^- a_p h_p) \\
&= \left(\frac{1}{\nu_p^-(B_{H_p^-}(r_1))\nu_p^a(B_{A_p}(r_2))} \right)^2 \int_{B_{H_p^-}(r_1)B_{A_p}(r_2)B_{H_p}(1)} |\Delta(a_p)^{-2}| \, d\mu_p(h_p^- a_p h_p) \\
&= \left(\frac{1}{\nu_p^-(B_{H_p^-}(r_1))\nu_p^a(B_{A_p}(r_2))} \right)^2 \int_{B_{H_p^-}(r_1)\times B_{A_p}(r_2)\times B_{H_p}(1)} |1| \, d\nu_p d\nu_p^a d\nu_p^- \\
&= \frac{1}{\nu_p^-(B_{H_p^-}(r_1))\nu_p^a(B_{A_p}(r_2))}.
\end{aligned}$$

So $||\phi||_2 = \left(\frac{1}{\nu_p^-(B_{H_p^-}(r_1))\nu_p^a(B_{A_p}(r_2))}\right)^{1/2} \leq (\frac{1}{\nu_p^-\otimes\nu_p^a(B_{\widetilde{H_p}}(r))})^{1/2}$ where r is smaller than the injective radius of the fixed point $a_n z_p$. □

Lemma 4. *The function ϕ defined above formula (6), is K_p-finite. The compact supported function f_p, which is invariant in the compact open subset W, is K_p-finite, i.e. denote the K_p-dimension as*

$$d_{K_p}(f_p) = \{\pi(k_p)f_p \mid k_p \in K_p\}.$$

And the K_p-dimension of functions ϕ and f are equal to group index $[SL_2(\mathbb{Z}_p) : W \cdot B_{H_p}(1)]$ and $[SL_2(\mathbb{Z}_p) : W]$ respectively, i.e.

$$d_{K_p}(\phi) = [SL_2(\mathbb{Z}_p) : W \cdot B_{H_p}(1)], \quad d_{K_p}(f_p) = [SL_2(\mathbb{Z}_p) : W].$$

Proof. As we know, $SL_2(\mathbb{Z}_p)$ is compact open and closed in $SL_2(\mathbb{Q}_p)$, so it can be covered finitely by W. Then we know that $[SL_2(\mathbb{Z}_p) : W]$ is finite. The same reason for $d_{K_p}(\phi)$, is that the product of two open subgroups is open. Since $W \subseteq WB_{H_p}(1)$, then $[SL_2(\mathbb{Z}_p) : WB_{H_p}(1)] \leq [SL_2(\mathbb{Z}_p) : W]$. □

Theorem 3.1 (Convergence rate of equidistribution of orbits over p-adic field)**.** *Given an irreducible unitary representation (π, H) of $G_\infty \times G_p$, for any $f \in S^W(X)$ as defined in 3, the effective bound of distribution of non-closed horocycle orbits is*

$$|A_{B_{H_p}(p^{2n})}(f, z_p) - \int_{X_p} f(x)\, d\overline{\mu_p}(x)| \ll d_W(\frac{1}{\nu_p(B_{\widetilde{H_p}}(r))})^{1/2}||f_p||_2\varpi(a_n).$$

Proof. For $f_p \in S(X_p)$, and taking $f_p = f_p - \frac{1}{\nu_p(B_{H_p})}\int_{X_p} f_p(x)\, d\overline{\mu_p}(x)$, the convergence of the equidistribution as follows

$$\frac{1}{\nu_p(B_{H_p})}\int_{B_{H_p}} f(h_p \cdot z_p))\, d\nu_p(h_p) - \int_{X_p} f(x)\, d\overline{\mu_p}(x),$$

can be turned into the estimate of the average

$$A_{B_{H_p}(p^{2n})}(f_p, z_p) = \frac{1}{\nu_p(B_{H_p})}\int_{B_{H_p}} f_p(h_p \cdot z_p)\, d\nu_p(h_p). \tag{7}$$

Then the matrix coefficient with function ϕ and f_p can be written as

$$\begin{aligned} < \phi, \pi(a_n)^{-1}f_p > &= \int_{X_p} \phi(x)f_p(a_n^{-1}x)\, d\overline{\mu_p}(x) \\ &= \int_{G_p} \phi(h_p^- a_p h_p a_n z_p) f_p(a_n^{-1}h_p^- a_p h_p a_n z_p)\, d\mu_p(h_p^- a_p h_p) \end{aligned}$$

By the product of the Haar measure, we write the average $A_{B_{H_p}}(f_p, z_p)$ as

$$A_{B_{H_p}}(f_p,z_p)=\int_{H_p^-}\int_{A_p}\int_{H_p}\theta^-(h_p^-)\theta^a(a_p)\chi_{B_{H_p}(1)}(h_p)f_p(a_n^{-1}h_pa_nz_p)\,d\nu_p\,d\nu_p^a\,d\nu_p^-$$
$$=\int_{G_p}\theta^-(h_p^-)\theta^a(a_p)\chi_{B_{H_p}(1)}(h_p)f_p(a_n^{-1}h_pa_nz_p)\Delta^{-1}(a_p)\,d\mu_p(h_p^-a_ph_p)$$

Then by our choice of ϕ, $A_{B_{H_p}}(f_p,z_p)$ can be approximated by the matrix coefficient,

$$|A_{B_{H_p}}(f_p,z_p)-<\phi,\pi(a_n)^{-1}f_p>|$$
$$=|\int_{G_p}\theta^-(h_p^-)\theta^a(a_p)\chi_{B_{H_p}(1)}(h_p)f_p(a_n^{-1}h_pa_nz_p)\Delta(a_p)^{-1}\,d\mu_p(h_p^-a_ph_p)-$$
$$\int_{G_p}\theta^-(h_p^-)\theta^a(a_p)\chi_{B_{H_p}(1)}(h_p)\Delta(a_p)^{-1}f_p(a_n^{-1}h_p^-a_ph_pa_nz_p)\,d\mu_p(h_p^-a_ph_p)|$$
$$=|\int_{G_p}\theta^-(h_p^-)\theta^a(a_p)\chi_{B_{H_p}(1)}(h_p)\Delta(a_p)^{-1}(f_p(a_n^{-1}h_pa_nz_p)-f_p(a_n^{-1}h_p^-a_pa_na_n^{-1}h_pa_nz_p))\,d\mu_p|$$

For the function f_p, we assume that it is W-invariant, where W contains $B_{\widetilde{H}_p}(r)$. So $A_{B_{H_p}}(f_p,z_p)=<\phi,\pi(a_n^{-1})f_p>$ inside of the injective radius ball $r(a_nz_p)$ due to the non-expanding conjugation lemma 1. The bound of $A_{B_{H_p}}(f_p,z_p)$ is obtained using the formula of the decay rate of matrix coefficient in the theorem 2.2, but we need to determine that ϕ and f_p are K_p-finite as the following lemma 4.
With the explicit K_p-dimension $d_{K_p}(\phi)$, $d_{K_p}(f_p)$ and estimation of norm of ϕ in lemma 3, the bound of our horocycle avaerage $A_{B_{H_p}}(f_p,z_p)$ is given by:

$$|A_{B_{H_p}}(f_p,z_p)-\int_{X_p}f(x)\,d\overline{\mu_p}(x)|\ll\sqrt{d_K(\phi)d_K(f_p)}||\phi||_2||f_p||_2\varpi(a_n),$$
$$=\sqrt{d_K(\phi)d_K(f_p)}(\frac{1}{Vol(B_{\widetilde{H}_p}(r))})^{1/2}||f_p||_2\varpi(a_n),$$
$$=[SL_2(\mathbb{Z}_p):W](\frac{1}{Vol(B_{\widetilde{H}_p}(r))})^{1/2}||f_p||_2\varpi(a_n).$$

□

Theorem 3.2. ***For general X, the theorem is not clear because this also needs more lemma about the height function.***

Proof. The horocycle average $A_B(f)$ can be rewritten by using the measure decomposition formulae,

$$\nu(B)A_B(f)(\Gamma(g_\infty,g_p))=\int_{B_\infty}\int_{B_p}f((h_\infty,h_p)\cdot\Gamma(g_\infty,g_p)\,d\nu_\infty(h_\infty)\,d\nu_p(h_p)$$
$$=\int_{G_\infty}\int_{G_p}\phi((h_\infty^-a_\infty h_\infty,h_p^-a_ph_p)\cdot\Gamma(g_\infty,g_p))$$
$$f((a_\infty^{'}h_\infty a_\infty^{'-1},a_n^{-1}h_pa_n)\cdot\Gamma(g_\infty,g_p)\,d\nu_\infty(h_\infty)\,d\nu_p(h_p),$$

where the function $\phi((h_\infty^- a_\infty h_\infty a_\infty^{'-1}, h_p^- a_p h_p a_n) \cdot \Gamma(g_\infty, g_p))$ is defined to take the value as

$$\theta_\infty^-(h_\infty^-)\theta_\infty^a(a_\infty)\chi_{B_\infty}(h_\infty)\Delta_\infty(a_\infty)^{-1}\theta_p^-(h_p^-)\theta_\infty^a(a_p)\chi_{B_p}(h_p)\Delta_p(a_p)^{-1}.$$

And the matrix coefficient $< \phi, \pi((a_\infty^{'-1}, a_n))^{-1} f >$ of the unitary representation $(\pi, \mathcal{H})$ of the product group $G_\infty \times G_p$ is shown as

$$\begin{aligned}
&\int_X \phi(z) \cdot \pi((a_\infty, a_p))^{-1} f(z)\, d\overline{\mu}(z) \\
&= \int_{G_\infty} \int_{G_p} \phi((h_\infty^- a_\infty h_\infty a_\infty^{'-1}, h_p^- a_p h_p a_n) \cdot \Gamma(g_\infty, g_p)) \pi((a_\infty^{'-1}, a_n))^{-1} \\
&\quad f((h_\infty^- a_\infty h_\infty a_\infty^{'-1}, h_p^- a_p h_p a_n) \cdot \Gamma(g_\infty, g_p))\, d\nu_\infty(h_\infty)\, d\nu_p(h_p) \\
&= \int_{G_\infty} \int_{G_p} \phi((h_\infty^- a_\infty h_\infty, h_p^- a_p h_p) \cdot \Gamma(g_\infty, g_p) \\
&\quad f((a_\infty^{'} h_\infty^- a_\infty h_\infty a_\infty^{'-1}, a_n^{-1} h_p^- a_p h_p a_n) \cdot \Gamma(g_\infty, g_p))\, d\nu_\infty(h_\infty)\, d\nu_p(h_p).
\end{aligned}$$

Then the average $A_B(f)$ can be approximated by the matrix coefficient $< \phi, \pi((a_\infty, a_p))^{-1} f >$:

$$\begin{aligned}
&|\nu(B) A_B(f)(\Gamma(g_\infty, g_p)) - < \phi, \pi((a_\infty, a_p))^{-1} f >| \\
&= |\int_{G_\infty} \int_{G_p} \phi((h_\infty^- a_\infty h_\infty, h_p^- a_p h_p) \cdot \Gamma(g_\infty, g_p)) \\
&f((a_\infty^{'} h_\infty a_\infty^{'-1}, a_n^{-1} h_p a_n)\, d\nu_\infty(h_\infty)\, d\nu_p(h_p) \\
&- \int_{G_\infty} \int_{G_p} \phi((h_\infty^- a_\infty h_\infty, h_p^- a_p h_p) \cdot \Gamma(g_\infty, g_p) \\
&f((a_\infty^{'} h_\infty^- a_\infty h_\infty a_\infty^{'-1}, a_n^{-1} h_p^- a_p h_p a_n) \cdot \Gamma(g_\infty, g_p))\, d\nu_\infty(h_\infty)\, d\nu_p(h_p). \\
&\le \int_{G_\infty} \int_{G_p} \phi((h_\infty^- a_\infty h_\infty, h_p^- a_p h_p) \cdot \Gamma(g_\infty, g_p)) \\
&|f((a_\infty^{'} h_\infty a_\infty^{'-1}, a_n^{-1} h_p a_n) - f((a_\infty^{'} h_\infty^- a_\infty h_\infty a_\infty^{'-1}, a_n^{-1} h_p^- a_p h_p a_n) \cdot \Gamma(g_\infty, g_p))|,
\end{aligned}$$

where by the non-expanding lemma 1,

$$\begin{aligned}
&f((a_\infty^{'} h_\infty^- a_\infty h_\infty a_\infty^{'-1}, a_n^{-1} h_p^- a_p h_p a_n) \\
&= f((a_\infty^{'} h_\infty^- a_\infty a_\infty^{'-1} a_\infty^{'} h_\infty a_\infty^{'-1}, a_n^{-1} h_p^- a_p a_n a_n^{-1} h_p a_n).
\end{aligned}$$

Continuing the calculation above, it will be bounded by

$$|\nu(B) A_B(f)(\Gamma(g_\infty, g_p)) - < \phi, \pi((a_\infty, a_p))^{-1} f >| \ll ||f||_{\mathrm{Lip}} \cdot ht(\Gamma(g_\infty, g_p)).$$

□