Contents

1 Introduction

Throughout this draft, the p-adic group is denoted as $G_p = SL_2(\mathbb{Q}_p)$, in which $\mathbb{Q}_p$ is the p-adic completion of the rational number $\mathbb{Q}$. Denote the p-adic integers as $\mathbb{Z}_p$, where all units are as $\mathbb{Z}_p^\times$. $\mathbb{R}$ is identified as $\mathbb{Q}_\infty$ and denotes $G_\infty = SL_2(\mathbb{R})$. Let Γ be a lattice of $G = G_\infty \times G_p$. We want to understand how the orbits of the analogue of horocyle flow distribute over the homogeneous space $X = \Gamma\backslash(G_\infty \times G_p)$, on which $G_\infty \times G_p$ acts by left translations, respectively, i.e. for a point $z \in X$, z can be written as $\Gamma(g_\infty^z, g_p^z)$, where (g_∞^z, g_p^z) is the representative of z. For any $(g_\infty, g_p) \in G$, the action is defined as

$$\Gamma(g_\infty^z, g_p^z) \cdot (g_\infty, g_p) = \Gamma(g_\infty^z g_\infty, g_p^z g_p).$$

Given each point z_∞ in the real homogeneous space $\Gamma'\backslash G_\infty$ where Γ' is a lattice of G_∞, the non-closed orbits starting from point z_∞ become asymptomatically equidistributed on $\Gamma'\backslash G_\infty$, as proven by Dani and Smillie [4, 5]. Later, around the early nineties, M. Ratner completed the uniform distribution for unipotent flow in a series of papers over real or S-adic Lie groups [13, 15, 14]. For this draft, we obtain the effective version of M. Ratner's results in the S-adic homogeneous space. In the interest of the effective version of the equidistribution problem, over the real group G_∞ with a cocompact lattice Γ', such estimates were obtained by Burger [1]. Furthermore, in the more general space $\Gamma'\backslash G_\infty$ that is non-compact but of finite volume, due to the existence of closed horocycle orbits, which are dense in the homogeneous space $\Gamma'\backslash G_\infty$, the rate of convergence of non-closed horocycle orbits is sensitively dependent on the starting point Γg_∞,which is known from Strömbergsson [16].

About the effective bound of the convergence rate of equidistribution of the non-closed orbits of horocycle flow in the space $X = \Gamma\backslash(G_\infty \times G_p)$, here is the basic statement about this problem: Let $\sigma : G \to X$ be the natural projection defined by $p(g) = \Gamma g$. For any fixed point $z \in X$, the projection $\sigma_z : G \to X$ is defined as $p_z(g) = zg$. On G_∞ or G_p, we have the usual Haar measures μ_∞ or μ_p induced from the corresponding invariant differential form. For the product topological group $G = G_\infty \times G_p$, the measure is defined by product: $\mu = \mu_\infty \otimes \mu_p$. Also denote by $\overline{\mu}$ the $G-$ invariant probability measure in X induced by μ, which is defined in the standard way.

Consider the subgroups of G_p, we define the maximal compact subgroup $SL_2(\mathbb{Z}_p)$ as K_p

and

$$A_p = \left\{ a_p(\omega, n) = \begin{pmatrix} \omega p^n & 0 \\ 0 & \omega^{-1}p^{-n} \end{pmatrix} \mid n \in \mathbb{Z}_{\geq 0} \text{ and } \ \omega \in \mathbb{Z}_p^\times \right\},$$
$$H_p = \left\{ h_p(t_p) = \begin{pmatrix} 1 & t_p \\ 0 & 1 \end{pmatrix} \mid t_p \in \mathbb{Q}_p \right\},$$
$$H_p^- = \left\{ h_p^-(t_p^-) = \begin{pmatrix} 1 & 0 \\ t_p^- & 1 \end{pmatrix} | t_p^- \in \mathbb{Q}_p \right\}.$$

For the elements in A_p, we shorten the notation as $a_n = a_p(1, n)$. And we omit the variables to shorten the notation to a_p, h_p, h_p^- if they are not needed in the context. Similarly, subgroups of G_∞ can be defined as follows: The maximal compact subgroup $SO(2, \mathbb{R})$ is denoted as K_∞,

$$\begin{aligned} K_\infty &= \{r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mid \theta \in \mathbb{R}\}, \\ A_\infty &= \left\{ a_\infty(x_\infty) = \begin{pmatrix} e^{x_\infty/2} & 0 \\ 0 & e^{-x_\infty/2} \end{pmatrix} \mid x_\infty \in \mathbb{R}_{\geq 0} \right\}, \\ H_\infty &= \left\{ h_\infty(t_\infty) = \begin{pmatrix} 1 & t_\infty \\ 0 & 1 \end{pmatrix} \mid t_\infty \in \mathbb{R} \right\}, \\ H_\infty^- &= \left\{ h_\infty^-(t_\infty^-) = \begin{pmatrix} 1 & 0 \\ t_\infty^- & 1 \end{pmatrix} \mid t_\infty^- \in \mathbb{R} \right\}. \end{aligned} \tag{1}$$

Fix Haar measures ν_∞ on H_∞ and ν_p on H_p, and ν on $H = H_\infty \times H_p$, given a function f defined on the homogeneous space X, We want to study ergodic average,

$$A_B(f)(z) = \frac{1}{\nu(B)} \int_B f((h_\infty, h_p) \cdot z) \, d\nu,$$

for $z = \Gamma(g_\infty, g_p) \in X$ that are not in any periodic $H_\infty \times H_p$ orbits, as the volume of $B = B_\infty(R_\infty) \times B_p(R_p)$ growing, that is defined as follow:

$$B_\infty(R_\infty) = \{h_\infty(t_\infty) \in H_\infty \mid |t_\infty|_\infty \leq R_\infty\},$$
$$B_p(R_p) = \left\{ h_p(t_p) \in H_p \mid |t_p|_p \leq R_p \right\}.$$

The goal is to obtain the effective bound of the convergence rate of this average. In the context of the product space X, a different methodology is required for the p-adic component compared to an isolated homogeneous space in the real case. In our first proof, the idea is a continuation of Margulis-Kleinbock thickening method [11]. The benefits of p-adic place is that the smooth functions give us local constancy, which means that our result is optimal in the sense of p-adic component. Here we used the Kunze-Stein inequality, which was first studied by Cowling [3] and generalized to homogeneous tree by Alessandro Veca [17]. Then combining with the well-known results for the real component by Burger and Strömbergsson [1, 16], the quantative equidistribution of non-closed horocycle orbits is shown in any non-compact but of finite volume homogeneous space X.

In order to avoid the loss from the thickening process, the second proof using representation by Burger is given.

Is there more history about padic work. Then explain the structure of this paper.

Other related work?

structure of this paper

In this paper, we want to study the quantitative estimate of the equidisbution of non-closed horocycle orbits on S-adic homogeneous space where S is a finite subset of all possible places. The first part is to estimate the distribution of non-closed orbits of horocycle flow on a p-adic homogeneous space by Margulis's thickening method [11]. In the second part with more abstract representation theory, we can study the quantative equidistribution of the horocycle flow in homogeneous space over $\mathbb{R} \times \mathbb{Q}_p$.

2 Representation theory of G

In this section, let us review the decay rate of the matrix coefficient of the unitary representation of G_p and the representation theory of the product group $G_\infty \times G_p$.

Definition 1 (Tempered representation). Given a unitary representation $(\pi, \mathcal{H}, G)$, it is called a tempered representation if there exists $\mathcal{V}$ dense in $\mathcal{H}$, for any $v \in \mathcal{V}$ and $\epsilon > 0$,

$$< \pi(g)v, v > \in L^{2+\epsilon}(G).$$

There is a rough classification of the irreducible unitary representations of the p-adic group G_p with respect to the decay property of matrix coefficients.

Theorem 2.1 (Classification of representations of G_p [10]). *The unitary irreducible representations of G_p, denoted by $(\pi, \mathcal{H})$, fall into the following three categories: the trivial representation 1_G, the tempered representations $(\pi, \mathcal{H})$, and the complementary series representations $(\gamma^s, \mathcal{H}_s)$ for $s \in (0, 1)$, i.e.*

$$\widehat{G_p} = \{1_G\} \cup \{\textit{temerped } \pi \in \widehat{G_p}\} \cup \{\gamma^s \mid s \in (0, 1)\}.$$

For the tempered representation defined above 1, its matrix coefficient has a better decay rate. For such a class of representations of the real or p-adic group, the matrix coefficient is bounded by the Harish-Chandra function in [2]. And the explicit expression of the p-adic Harish-Chandra function is shown in theorem 2.2. With the classification of the representations of G_p, every irreducible representation corresponds to a spherical function. And there is an explicit asymptotic formula for the spherical function ϖ in each case.

Definition 2 (Harish-Chandra function)**.**

Definition 3 (Spherical function)**.**

Theorem 2.2 (Asymptotic formulae of spherical functions [8])**.** *If the representation (π, H_π) of G_p is tempered, then the Harish-Chandra function is*

$$\Xi(a_n) = \frac{p^{-n}}{p+1}((2n+1)(p-1)+2).$$

Similarly, for the complementary series representation (γ^s, H_s) of G_p, the spherical function has the following bound,

$$\varpi(a_n) \ll \frac{p^{n(s-1)}}{1-p^{-s}}.$$

Theorem 2.3 (Decay rate of matrix coefficient of representation of $G_p = SL_2(\mathbb{Q}_p)$)**.** *For a irreducible unitary representation $(\pi, \mathcal{H})$ of G_p, then for any K_p-finite vector ψ, ϕ in $\mathcal{H}$, such that*

$$< \pi(a_n)\phi, \psi > \leq \sqrt{d_K(\phi)d_K(\psi)}||\phi||_2||\psi||_2\varpi_\pi(a_n).$$

in which $a_n \in A_p$ and ϖ_π is denoted as the spherical function corresponding to the representation π.

For the unitary representation of the group $G_\infty \times G_p$, we need to introduce some basic concepts and definitions. The study of unitary representation of G_∞ is dependent on Bargmann's classification; more details can be found in [7]. The expression of a p-adic representation is similar to the real part.
To study the general unitary representation, here is a short review of the direct integral decomposition of unitary representation. The discussion begins with the unitary representation of G_∞. Let $(Z_\infty, \mu_{Z_\infty})$ be a Borel measure space, for each $\xi_\infty \in Z_\infty$, $(\pi_{\infty,\xi_\infty}, \mathcal{H}_{\infty,\xi_\infty})$ is an irreducible unitary representation of G_∞. The direct integral

$$\pi_\infty = \int_{Z_\infty}^{\oplus} \pi_{\infty,\xi_\infty}\, d\mu_{Z_\infty}(\xi_\infty), \mathcal{H}_\infty = \int_{Z_\infty}^{\oplus} \mathcal{H}_{\infty,\xi_\infty}\, d\mu_{Z_\infty}(\xi_\infty)$$

is defined as a unitary representation $(\pi_\infty, \mathcal{H}_\infty)$ of G_∞. Fixed the basis

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2}$$

of the corresponding Lie algebra $\mathfrak{g}_\infty$ of G_∞. And the Casimir operator is defined for the C^2-vectors in the Hilbert space $\mathcal{H}_\infty$ by the formula

$$\Omega_\infty = -\frac{1}{4}(Y^2 + 2X_+X_- + 2X_-X_+).$$

And let $\Delta = -\sum_i X_i^2 \in U(\mathfrak{g}_\infty)$ where $\{X_i\}$ is any fixed basis of $\mathfrak{g}_\infty$. Then the closure $\overline{\pi_\infty(\Delta)}$ of $\pi_\infty(\Delta)$ is a self-adjoint operator on $\mathcal{H}_\infty$ [12].
Given an irreducible unitary representation $(\pi_{\infty,\xi_\infty}, \mathcal{H}_{\infty,\xi_\infty})$ of G_∞ where $\mathcal{H}_\infty$ is a separable complex Hilbert space with basis

$$\Phi_\infty(\xi_\infty) = \{\phi^{(n)}_{\infty,\xi_\infty} = \phi^{(n)}_\infty(\xi_\infty) \mid n \in \Sigma \subseteq 2\mathbb{Z}\}.$$

In this case that π_{∞,ξ_∞} is irreducible, the Casimir operator $\pi_{\infty,\xi_\infty}(\Omega_\infty) = \lambda id$ acts as a scalar on the space of smooth vectors $\mathcal{H}^\infty_{\infty,\xi_\infty}$.
According to Bargmann's classification of irreducible unitary representations of G_∞,

1. when $\lambda > 0, \Sigma = 2\mathbb{Z}$, the irreducible representation $(\pi_{\infty,\xi_\infty})$ with ξ_∞ is a principal series representation or a complementary representation.

2. when $\lambda = 0, \Sigma = \{0\}$, it is just a trivial representation.

3. when $\lambda = \frac{m}{2}\left(1 - \frac{m}{2}\right)$, $\Sigma = \{m, m+2, m+4, \dots\}$ or $\{-m, -m-2, -m-4, \dots\}$, it is a discrete series representation.

Now adopting the information of irreducible representations in decomposition, for almost all $\xi \in Z$, $\lambda = \lambda(\xi)$, the Casimir operator

$$\pi_{\infty,\xi}(\Omega_\infty) = \lambda(\xi) id$$

in $H_{\infty,\xi}$ which has a basis $\{\phi^{(n)}_{\infty,\xi} = \phi^{(n)}_\infty(\xi) \mid n \in \mathbb{Z}\}$. And these bases can be chosen as $K_\infty-$eigenvectors, i.e.

$$\pi_{\infty,\xi}(r(\theta))\phi^{(n)}_{\infty,\xi} = e^{in\theta}\phi^{(n)}_{\infty,\xi}.$$

The eigenvalue λ can be parameterized as follows, for a unique number $s = s(\xi_\infty)$ such that $\lambda = s(1-s)$ and $\mathrm{Re}(s) \geq \frac{1}{2}, \mathrm{Im}(s) \geq 0$. Then $s \in \frac{1}{2} + i\mathbb{R}$ if $\lambda \geq \frac{1}{4}$, $s \in (\frac{1}{2}, 1]$ if $0 \leq \lambda < \frac{1}{4}$, $s \in \mathbb{Z}^+$ if $\lambda < 0$.

1. $Z^0_\infty \cup Z^{rs}_\infty$ is discrete measure spaces, for trivial $\pi_{\infty,\xi} = 1$, $\mu_{Z_\infty}(\{\xi_\infty\}) = 1$, for any $\xi_\infty \in Z^0_\infty \cup Z^{rs}_\infty$,

2. Z^{rs}_∞ is finite. And there exists unique $\xi \in Z^{rs}$, such that $\pi_{\infty,\xi}$ is trivial. For all the other $\xi \in Z^{rs}$, we have $s(\xi) \in (\frac{1}{2}, 1)$,

3. For any $\xi \in Z^{ct}_\infty$, $\pi_{\infty,\xi}$ is principal series representation and $s(\xi) \in \frac{1}{2} + i\mathbb{R}_{\geq 0}$.

Then given a Borel measure (Z, μ), the unitary representation $(\pi, \mathcal{H})$ of the product group $G_\infty \times G_p$ can be written as direct integral

$$\pi = \int_Z^\oplus \pi_{\infty,\xi} \otimes \pi_{p,\xi} \, d\mu_Z(\xi),$$

and the representation space is the Hilbert space $\mathcal{H}$ with direct integral decomposition

$$\int_Z^\oplus \mathcal{H}_{\infty,\xi} \otimes \mathcal{H}_{p,\xi} \, d\mu_Z(\xi).$$

Now, in the direct integral $\mathcal{H} = \int_Z^\oplus \mathcal{H}_{\infty,\xi} \otimes H_{p,\xi} \, d\mu(\xi)$, the $\mathcal{H}_\xi = H_{\infty,\xi} \otimes H_{p,\xi}$ can be further classified only with respect to the spectrum decomposition of $H_{\infty,i}$ to the disjoint union of $Z = Z^0 \cup Z^{rs} \cup Z^{ct}$.

For each irreducible representation $\mathcal{H}_{\infty,\xi_\infty} \otimes \mathcal{H}_{p,\xi}$, there are basis defined as

$$\Phi_\infty(\xi_\infty) = \{\phi^{(n)}_{\infty,\xi} = \phi^{(n)}_\infty(\xi) \mid n \in \mathbb{Z}\},$$
$$\Phi_p(\xi) = \{\phi^{(m)}_{p,\xi} = \phi^{(m)}_p(\xi) \mid n \in \mathbb{Z}\}.$$

And the Hilbert space $\mathcal{H}_\infty \otimes \mathcal{H}_p$ contains all the functions $f_\infty \otimes f_p$ defined on Z. In order to make the definitions clearer, we use those pure tensor product functions to express the formulae. The functions $f_\infty(\xi) \otimes f_p(\xi)$ belong to the irreducible component $\mathcal{H}_{\infty,\xi} \otimes \mathcal{H}_{p,\xi}$ which are $\Phi_\infty \otimes \Phi_p$-measurable, that is, $< f_\infty(\xi), \phi^{(n)}_{\infty,\xi} > \cdot < f_p(\xi), \phi^{(m)}_{p,\xi} >$ is a measurable function of (ξ, ξ) for all $n, m \in \mathbb{Z}$. And all the $\Phi_\infty \otimes \Phi_p$-measurable functions satisfy that

$$\int_Z ||f_\infty(\xi)||^2 ||f_p(\xi)||^2 \, d\mu_Z(\xi) < \infty.$$

We also assume that there exists a small eigenvalue λ_1 in the spectrum of the Casimir operator. And define s_1 by $\lambda_1 = s_1(1 - s_1)$.

Definition 4 (decay expoenent of irreducible representation)**.** Let $s \in (0, 1)$. The complementary series γ^s has an almost decay exponent

$$\kappa_{\gamma^s} = 1 - s$$

where s^2 is the eigenvalue of the Casimir operator. And the decay exponent of other tempered irreducible representation is defined to be 1.

For a unitary representation $(\pi, \mathcal{H})$ of $G_\infty \times G_p$ with direct integral decomposition $\pi = \int_Z \pi_{\infty,\xi} \otimes \pi_{p,\xi} \, d\mu_Z(\xi)$, there exist decay exponents

$$\kappa_{\infty,\xi} = 1 - s_\infty(\xi), \;\; \kappa_{p,\xi} = 1 - s_p(\xi),$$

for the irreducible family $\pi_{\infty,\xi}$ and $\pi_{p,\xi}$.
The matrix coefficient $< \pi(g)v, w >$ for $g = (g_\infty, g_p)$ and K-finite vectors $v, w \in \mathcal{H}$,

$$\begin{aligned}
&< \int_Z \pi_{\infty,\xi} \otimes \pi_{p,\xi}(g) \, d\mu_Z(\xi) \cdot v, w > \\
&= \int_Z < \pi_{\infty,\xi}(g_\infty) v_\xi, w_\xi > \cdot < \pi_{p,\xi}(g_p v_\xi, w_\xi) > \, d\mu_Z(\xi)
\end{aligned}$$

For the general unitary representation $(\pi, \mathcal{H}, G_\infty \times G_p)$, the following theorem shows the effective decay rate of its matrix coefficient [6].

Theorem 2.4. *Given a unitary representation* $(\pi, \mathcal{H})$*, assume that there exists* $\kappa_\infty = min(1 - s(\xi))$ *for the complementary series representations in the irreducible family* $\pi_{\infty,\xi}$ *and* $\kappa_p = min(1 - s(\xi))$ *same for the complementary series representations in the irreducible family* $\pi_{p,\xi}$*. Then for any* $K_\infty \times K_p$*-finite vectors, we have*

$$|< \pi((a_n, a_t))v, w >| \leq \sqrt{d_{K_\infty(v)} d_{k_\infty}(w) d_{K_p}(v) d_{K_p}(w)} ||v|| ||w|| \Xi_\infty(a_t)^{\kappa_\infty} \Xi_p(g)^{\kappa_p},$$

where Ξ_∞ *and* Ξ_p *are the Harish-Chandra functions.*

3 Sobolev Embedding theorems and height

Other very important ingredients that are used later in the proof are Sobolev embedding theorems, which require a careful estimation of the injective radius of the specific point. To study the injective radius, in this section, we define the height function of any point $\Gamma(g_\infty, g_p) \in X$.

Definition 5. Consider the Separable Hilbert space $\mathcal{H}_\infty \otimes \mathcal{H}_p$ of functions $f_\infty \otimes f_p$ defined on $G_\infty \times G_p$, and the k-Sobolev norm W_k is defined with respect to the basis of Lie algebra in

$$||f_\infty \otimes f_p||_{W_k} = ||(I + \overline{\pi_\infty(\Delta)})^{k/2} f_\infty|| \cdot ||f_p||_2$$

The space of smooth vectors H^∞ is dense in the space $W_k(H_\infty \otimes H_p)$ of finite Sobolev norm.

The equivalent formulation of Sobolev norm above is

$$||f_\infty \otimes f_p||_{W^k}^2 = \int_Z \sum_{n\in\Sigma(\xi)} (1+n^2+|s(\xi)|^2)^k |< f_{\infty,\xi}, \phi_{\infty,\xi}^n >_{\mathcal{H}_\xi}|^2 \, d\mu(\xi) \cdot ||f_p||_2^2.$$

We consider the composition of the Hilbert space with respect to the classification of a real unitary representation.

Lemma 1. *Denote $Z_+^0 = \left\{\xi \in Z^0 \mid s(\xi) \in \left(\frac{1}{2}, 1\right)\right\}$. And $\mathcal{H}_-^0$ is defined to be an orthogonal component of $\mathcal{H}_+^0$ that corresponds to Z_+^0. The other closed subspaces corresponding to Z^{rs} and Z^{ct} are denoted $\mathcal{H}^{rs}$ and $\mathcal{H}^{ct}$, respectively.*

So for $f \in \mathcal{H}^\infty$, the Sobolev norm of f can be calculated as

$$||f||_{W_k}^2 = ||f_0||_{W_k}^2 + ||f_{ct}||_{W_k}^2 + ||f_{rs}||_{W_k}^2.$$

since $\mathcal{H}^0$, $\mathcal{H}^{ct}$ and $\mathcal{H}^{rs}$ are all closed Hilbert subspaces.
Now we can start to introduce the Sobolev embedding theorem, beginning with the injective radius. For the homogeneous space $X = \Gamma\backslash G$ where Γ is of finite covolume and $G = G_\infty \times G_p$, we recall the definition of injective radius: The distance d_G is defined as

$$d_G((g_\infty, g_p), (Id_\infty, Id_p)) = \max\{d_{G_\infty}(g_\infty, Id_\infty), d_G(g_p, Id_p)\},$$

and the distance d_X is defined as

$$d_X(\Gamma(g_\infty^o, g_p^o), \Gamma(g_\infty, g_p)) = \inf_{\gamma\in\Gamma} \max\{d_{G_\infty}(g_\infty, \gamma g_\infty^o), d_G(g_p, \gamma g_p^o)\}.$$

Definition 6. For any $\Gamma(g_\infty^o, g_p^o) \in X$, there exists some $r > 0$ such that the map from

$$B_r^G = \{g = (g_\infty, g_p) \in G \mid d_G((g_\infty, g_p), (Id_\infty, Id_p)) < r\}$$

to

$$B_r^X(\Gamma(g_\infty^o, g_p^o)) = \{\Gamma(g_\infty, g_p) \mid d_X(\Gamma(g_\infty^o, g_p^o), \Gamma(g_\infty, g_p)) < r\}$$

defined by $g \to \Gamma(g_\infty^o, g_p^o)g$ is an isometry.

To estimate the behavior of the injective radius in homogeneous space X, we define a height function

$$ht(\Gamma(g_\infty, g_p)) = \sup\nolimits_{\gamma\in\Gamma_{g_p}} \left\{\frac{1}{\operatorname{Im}(\gamma z)} \mid z = g_\infty(i) \in \mathbb{H}\right\},$$

where $\Gamma_{g_p} = \{\gamma \in \Gamma \mid \gamma g_p \in SL_2(\mathbb{Z}_p)\}$. By restricting the p-adic component to the compact ball $SL_2(\mathbb{Z}_p)$, the height of the point $\Gamma(g_\infty, g_p)$ is controlled by the real component.

Lemma 2. *Given a fixed point $x_0 \in X$, there exists a positive constant depending on the lattice Γ, the fixed point x_0 and the g_p, such that for any point $\Gamma(g_\infty, g_p)$,*

$$ht(\Gamma(g_\infty, g_p)) \asymp e^{d_X(\Gamma(g_\infty,g_p),x_0)}. \tag{3}$$

Proof. This needs to be done. $\square$

Lemma 3 (Sobolev Embedding theorem [9]). *If $F \in W^2(H)$ then f is a continuous function on X, and for any point $(\Gamma(g_\infty, g_p)) \in X$,*

$$|F(\Gamma(g_\infty, g_p))| \ll_\Gamma ht(\Gamma(g_{\infty,g_p}))^{1/2}\mu_p(W)^{-1/2}||F||_{W^{3,2}}$$

Proof. By the classical results of Sobolev embedding theorem over the upper half plane, we know that for any $G \in W^2(H)$ and invariant under the action of open compact subgroup W of $SL_2(\mathbb{Q}_p)$, then for fixed g_p, G is continuous with respect to the real component. And there exists a constant $C(\epsilon, g_p) > 0$, for any $g_\infty \in G_\infty$,

$$|G(\Gamma(g_\infty, g_p))|^2 < C(\epsilon)\int_{B(\Gamma(g_\infty,g_p),\epsilon)} |G(y)|^2 + |dG(y)|^2 + |\Delta G(y)|^2\, dy,$$

where $B(\Gamma(g_\infty, g_p), \epsilon) = \{\Gamma(g'_\infty, g_p)|d_X(\Gamma(g_\infty, g_p), \Gamma(g'_\infty, g_p)) \le \epsilon\}$.
From now on, in the discussion, we consider the fixed g_p. For a point $\Gamma(g_\infty, g_p)$ in the space X, denote by $\rho(\Gamma(g_\infty, g_p))$ the radius of injectivity at the point $\Gamma(g_\infty, g_p)$.
Let $\epsilon \ll \pi$, denote by $A_0 = \{\Gamma(g_\infty, g_p) \in X \mid \rho(\Gamma(g_\infty, g_p)) > \epsilon\}$ the open subset. When the injective radius is chosen sufficiently small, the complement A_0^c consists of k connected components, which are contained in T^1A_i, the unit tangent bundle of disjoint open cusps. And those connected open subset around cusps have a horocycle orbit of length 2ϵ as their boundary.
Then by the Sobolev embedding theorem stated above, there exists a constant $C(\epsilon, g_p) > 0$, such that for any point $\Gamma(g_\infty, g_p) \in A_0$, we have

$$|F(\Gamma(g_\infty, g_p))|^2 < C(\epsilon, g_p)\int_{B(\Gamma(g_\infty,g_p),\epsilon)} |F(y)|^2 + |dF(y)|^2 + |\Delta F(y)|^2\, dy.$$

Let $\tilde{F}$ be the lift of the function F to the space $G_\infty \times G_p/K_\infty \times K_p$, and by denoting $x = \Gamma(g_\infty, g_p)$, $\tilde{x}$ is denoted as the elements from the preimage of $x \in X$ under the projection

$$G_\infty \times G_p \to \Gamma\backslash(G_\infty \times G_p)$$

Then

$$|\tilde{F}(\Gamma(g_\infty, g_p))|^2 < C(\epsilon, g_p)\int_{B(\tilde{x},\epsilon)} |\tilde{F}(y)|^2 + |d\tilde{F}(y)|^2 + |\Delta\tilde{F}(y)|^2\, dy.$$

By the property of height of the point $\Gamma(g_\infty, g_p)$ above 3, we know that the ball $B(\tilde{x}, \epsilon) \subset G_\infty \times G_p$ covers the ball $B(\Gamma(g_\infty, g_p), \epsilon)$ less than $[C'(\epsilon, g_p) ht(\Gamma(g_\infty, g_p))] + 1$ times where obviously the height $ht(\Gamma(g_\infty, g_p))$ is bounded on A_0.
In conclusion, we get

$$|F(\Gamma(g_\infty, g_p))| \ll ht(\Gamma(g_\infty, g_p))^{1/2} ||F(\Gamma(\cdot, g_p))||_{W^{3,2}},$$

where $||F(\Gamma(\cdot, g_p))||$ is meant to be Sobolev norm of real component with fixed g_p.
Finally, we can show the complete Sobolev norm $||F||_{W^{3,2}}$,

$$\begin{aligned}
||F||^2_{W^{3,2}} &= \int_{G_p/W} \int_{W g_p} ||F(\Gamma(\cdot, w g_p))||^2_{W^{3,2}}\, dw \\
&\geq \int_{W g_p} ||F(\cdot, w g_p)||^2_{W^{3,2}}\, dw \\
&= ||F(\Gamma(\cdot, g_p))||^2_{W^{3,2}} \cdot \mu_p(W)
\end{aligned}$$

In conclusion,

$$||F(\Gamma(g_\infty, g_p))|| \ll_\Gamma ht(\Gamma(g_{\infty, g_p}))^{1/2} \mu_p(W)^{-1/2} ||F||_{W^{3,2}}.$$

$\square$

As in the same philosophy, we can show the other proofs about the Sobolev embedding theorems for different functions corresponding to different Casirmir eigenvalues in the spectrum decomposition of the representation.

Lemma 4 (Sobolev embedding theorem for cuspidal functions [16])**.** *If $F \in W^3(\mathcal{H})$, and $F \in \mathcal{H}_0$ is cuspidal then the function F is pointwise bounded by*

$$|F(\Gamma(g_\infty, g_p))| \ll_\Gamma ||F||_{W^3} \cdot ht(\Gamma(g_\infty, g_p))^{-1/2} \mu_p(W)^{-1/2}.$$

Lemma 5 (Sobolev embedding theorem for the residual eigenfuntion [16])**.** *And Let $\xi \in Z^{rs}$, and let $s = s(\xi)$ be the parameter of eigenvalue λ as defined above, then we have for every $F \in W^3(\mathcal{H}) \cap \mathcal{H}_\xi$ and any point $\Gamma(g_\infty, g_p) \in X$,*

$$|F(\Gamma(g_\infty, g_p))| \ll_\Gamma ||F||_{W^3} \cdot ht(\Gamma(g_\infty, g_p))^{1-s} \cdot \mu_p(W)^{-1/2}$$

4 Effective equidistribution of horocycle flow over the homogenous space X

4.1 Statement of the main theorems

The theorem, which will be stated and demonstrated in this Section, aims to establish the effective equidistribution of non-closed horocycle orbits over space $X = \Gamma\backslash(G_\infty \times G_p)$. The ergodic average of horocycle orbits starting from $z_\infty \otimes z_p$ is defined as

$$A_B(f,(z_\infty, z_p)) = \frac{1}{\nu(B)} \int_B (\pi_\infty(h_\infty) \otimes \pi_p(h_p))(f_\infty \otimes f_p)(z_\infty, z_p)\, d\nu_\infty(h_\infty) d\nu_p(h_p),$$

where f is an element of the space $\mathcal{H} = L^2(X)$, and the measure $\nu(B)$ is specified by $\nu_\infty(B_\infty(r))\nu_p(B_p(p^n))$.

Theorem 4.1 (Convergence rate of equidistribution of orbits over real and p-adic field)**.** *For a unitary representation (π, H) of the group $G_\infty \times G_p$ with the smallest eigenvalue λ_1,*

The theorem needs more work which is similar to Strombergsson's paper. We need to deal with some kinda conditions on the starting point.

4.2 Proof of the theorem 4.1

Proof. Here we recall the decomposition of the representation space $\mathcal{H}$ from the above lemma 1,

$$\mathcal{H} = \int_Z \mathcal{H}_\xi \, d\mu_Z(\xi),$$

therefore the theorem is proved by considering $\xi \in Z^0_+ \cup Z^{rs}$, $\xi \in Z^0_-$ and $\xi \in Z^{ct}$. The smooth vectors $\mathcal{H}^\infty \subset \mathcal{H}$ are dense, so, with loss of generality, we can assume that all our functions are smooth.

With an orthogonal basis $\{\phi^{(n)}_{\infty,\xi} = \phi^{(n)}_\infty(\xi) \mid n \in \mathbb{Z}\}$, of the Hilbert space $\mathcal{H}_\xi$, the vector $f \in \mathcal{H}_\xi$ can be written

$$f = \sum_{n \in \Sigma(\xi)} c_{n,\xi} \phi^{(n)}_{\infty,\xi} \otimes f^{(n)}_{p,\xi}. \tag{4}$$

And Denote $A_B(f)(\Gamma(g_\infty, g_p))$ by the horocycle average

$$\frac{1}{\nu(B)} \int_B f((h_\infty, h_p) \cdot \Gamma(g_\infty, g_p))\, d\nu_\infty(h_\infty)\nu_p(h_p).$$

Since

$$
\begin{aligned}
&|\frac{1}{\nu(B)}\int_B f((h_\infty,h_p)\cdot\Gamma(g_\infty,g_p))\,d\nu_\infty(h_\infty)\nu_p(h_p)| \\
&\leq \frac{1}{\nu(B)}\int_B |f((h_\infty,h_p)\cdot\Gamma(g_\infty,g_p))|\,d\nu_\infty(h_\infty)\nu_p(h_p),
\end{aligned}
$$

then the Sobolev embedding theorems above can be plugged in here to control $|f((h_\infty,h_p)\cdot\Gamma(g_\infty,g_p))|$ by Sobolev norms.

A lot of words is missing, more details are needed to explain those formulae.

And another very important ingredient of our proof is the key lemma from Burger [1]: Fix a number $\mathcal{Y}\geq 1$. For each $y\geq 1$, we define F_y and S_y to be the intertwining operators $\mathbb{H}\to\mathbb{H}$ which are determined, via the integral decomposition of $\mathcal{H}$, by the following functions $Z\to\mathbb{C}$,

$$
f_y(\xi)=\begin{cases}\frac{sy^{s-1}-(1-s)y^{-s}}{2s-1}, & \text{if } \mathrm{Re}(s)<1, s\neq\frac{1}{2},\\ \frac{2+\log y}{2\sqrt{y}}, & \text{if } s=\frac{1}{2},\\ y^{-s}, & \text{if } s\in\mathbb{Z}^+,\end{cases} \tag{5}
$$

$$
s_y(\xi)=\begin{cases}\frac{y^{s-1}-y^{-s}}{2s-1}, & \text{if } \mathrm{Re}(s)<1, s\neq\frac{1}{2},\\ \frac{\log y}{2\sqrt{y}}, & \text{if } s=\frac{1}{2},\\ 0, & \text{if } s\in\mathbb{Z}^+,\end{cases} \tag{6}
$$

And also, for each $y>0$, one more intertwining operator $T_y:\mathcal{H}\to\mathcal{H}$ is determined by

$$
t_y(\xi)=\begin{cases}s_y(\xi), & \text{if } \mathrm{Re}(s)<1, y\geq 1,\\ 0, & \text{if } \mathrm{Re} s<1, y<1,\\ y^{s-1}\left(\frac{\max(1,y)^{1-2s}-\mathcal{Y}^{1-2s}}{1-2s}\right), & \text{if } s\in\mathbb{Z}^+.\end{cases} \tag{7}
$$

The bounds of such functions are obtained by Buger [1],

$$
\sup_{\xi\in Z^0_-\cup Z^{ct}}|f_y(\xi)|\leq\frac{2+\log y}{2\sqrt{y}},\ \sup_{\xi\in Z^0_-\cup Z^{ct}}|s_y(\xi)|\leq\frac{\log y}{\sqrt{y}},\quad\text{for any } y\geq 1, \tag{8}
$$

$$
\sup_{\xi\in Z^0_-\cup Z^{ct}}|t_y(\xi)|\leq\frac{2\log(y+10)}{\sqrt{y+1}},\ \text{for any } y\in(0,\mathcal{Y})]. \tag{9}
$$

In view of the estimates established in the equation above and the finiteness of the sets Z^0_+ and Z^{rs}, we deduce that the operators F_y, S_y, and T_y are bounded from $\mathcal{H}$ to $\mathcal{H}$ for each fixed y. Furthermore, for each $k>0$, these operators are bounded from $W_k(\mathcal{H})$ to $W_k(\mathcal{H})$ and are continuous from $\mathcal{H}^\infty$ into $\mathcal{H}^\infty$. The continuity of $t_y(\zeta)$ with respect to y, uniformly in ζ, can be uniformly bounded; that is to say, for each prescribed $y_0>0$,

$$
\sup_{\zeta\in Z}|t_y(\zeta)-t_{y_0}(\zeta)|\to 0 \text{ as } y\to y_0.
$$

As a consequence of this, the operator T_y is continuous in y with respect to the operator norm within each space $W_k(\mathcal{H})$, where $k > 0$, with

$$\|T_y - T_{y_0}\|_{W_k} \to 0 \text{ as } y \to y_0.$$

The corresponding integral representation is as follows: Given a unitary representation $(\pi, \mathcal{H}, SL_2(\mathbb{R}))$, for any $\mathcal{Y} \geq 1$ and $T > 0$, we have

$$\begin{aligned} \frac{1}{T}\int_0^T \pi(h(\infty))f\,dt = & \frac{1}{T}\int_0^T \pi(h_\infty a_\infty(\mathcal{Y}))F_\mathcal{Y}(f)\,dt \\ & - \frac{1}{2T}\int_0^T \pi(h_\infty a_\infty(\mathcal{Y}))S_\mathcal{Y}(d\pi(Y))f\,dt \\ & + \frac{1}{T}\int_o^\mathcal{Y} [1 - \pi(h_\infty(T))]\pi(a_\infty(y))T_y(d\pi(X_-)f)\,dy. \end{aligned} \tag{10}$$

where $h_\infty = h_\infty(t)$, and the definition of h_∞ and a_∞ are from 1. And to adopt this equation in the S -adic setting, the derivative Lie with respect to the unitary representation $(\pi, \mathcal{H}, G_\infty \times G_p)$ is defined as follows, for any $f \in \mathcal{H}$ with the chosen basis of Lie algebra in 2,

$$d\pi(Y)f = \lim_{t\to 0} \frac{\pi(\exp(tY), Id_p)f - f}{t}.$$

So when $\xi \in Z^{ct}$ such that $f \in \mathcal{H}^{ct}$, in this case for any $\xi \in Z_{ct}$, $s \in \frac{1}{2} + i\mathbb{R}$. We may here replace the integral $\int_0^\mathcal{Y}$ in the last line by $\int_1^\mathcal{Y}$.For every fixed point $\Gamma(g_\infty, g_p) \in X$, the map $f \mapsto f(\Gamma(g_\infty, g_p))$ is a linear functional on $\mathcal{H}^\infty$.Then the lemma of Burger above can be also rewritten in the format of functions with point-wise evaluation, given a unitary representation $(\pi, H, G_\infty \times G_p)$, for a function $f \in H$ and any point $\Gamma(g_\infty, g_p)$,

$$\begin{aligned} & \int_{B_p}\int_{B_\infty} f((h_\infty, h_p)\Gamma(g_\infty, g_p))\,d\nu_\infty(h_\infty)\nu_p(h_p) \\ & = \int_{B_p}\int_{B_\infty} [F_\mathcal{Y} f]((h_\infty a_\mathcal{Y}, h_p)\Gamma(g_\infty, g_p))\,d\nu_\infty(h_\infty)\nu_p(h_p) \\ & - \frac{1}{2}\int_{B_p}\int_{B_\infty} d[S_\mathcal{Y} d\pi(Y)f]((h_\infty a_\infty(\mathcal{Y}), h_p)\Gamma(g_\infty, g_p))d\nu_\infty(h_\infty)\nu_p(h_p) \\ & + \int_{B_p}\int_1^Y ([T_y d\pi(X_-)]f((a_\infty(y), h_p)\Gamma(g_\infty, g_p)) \\ & - [T_y d\pi(X_-)f]((h_\infty(T)a_\infty(y)), h_p)\Gamma(g_\infty, g_p))\,dyd\nu_p(h_p). \end{aligned} \tag{11}$$

The first three terms constitute integrals over the interval $[0, T]$ with respect to the $\mathcal{H}^\infty$-valued integrands, which are continuous mappings into $\mathcal{H}^\infty$. The continuity of these mappings guarantees the well-defined nature of the integrals in the Banach space $\mathcal{H}^\infty$. In a similar manner, the integrand corresponding to the final term is a continuous mapping from $(0,\mathcal{Y}]$ into $\mathcal{H}^\infty$. By considering the truncated integral $\int_\epsilon^{\mathcal{Y}}$, for an arbitrary $\epsilon > 0$, we ensure the well-definedness of the final integral within $\mathcal{H}^\infty$. Moreover, as ϵ tends to zero, the convergence of this integral within $\mathcal{H}$ is guaranteed by the uniform boundedness of the integrand's norm over the interval $(0,\mathcal{Y}]$, with respect to the norm denoted by $\|\cdot\|$. Let's consider the first case, when $\xi \in Z^{ct}$ such that $f \in \mathcal{H}^{ct}$, by using the lemma, we have

$$
\begin{aligned}
|A_B(f)(\Gamma(g_\infty, g_p))| \ll_\Gamma & \frac{1}{\nu(B)\mu_p(W)^{1/2}} \\
& \times \int_B ||\pi((h_\infty, h_p))f||_{W_3} \cdot ht(\Gamma(g_\infty, g_p))^{1/2}\, d\nu_\infty(h_\infty)\nu_p(h_p)
\end{aligned}
$$

Plugging the formula of expansion of f above,

$$
\begin{aligned}
& \int_B ||\pi((h_\infty, h_p))f||_{W_3}\, d\nu_\infty(h_\infty)\nu_p(h_p) \\
= & \int_B \left|\left| \int_{Z^{ct}} \sum_{n\in\Sigma(\xi)} c_{n,\xi}\pi_{\infty,\xi}(h_\infty)\cdot \phi^{(n)}_{\infty,\xi} \otimes \pi_{p,\xi}(h_p) f^{(n)}_{p,\xi})\, d\mu_Z(\xi) \right|\right|_{W_3} d\nu_\infty(h_\infty)\nu_p(h_p) \\
\leq & \int_{Z^{ct}} \sum_{n\in\Sigma(\xi)} |c_{n,\xi}|^2 \int_B ||\pi_{\infty,\xi}(h_\infty)\cdot\phi^{(n)}_{\infty,\xi}||_{W_3} ||\pi_{p,\xi}(h_p) f^{(n)}_{p,\xi})||_2\, d\nu_\infty(h_\infty)\nu_p(h_p)\, d\mu_Z(\xi)
\end{aligned}
$$

And from the results 8,9 above, we also have $\|F_Y f\|_{W_3} \leq \frac{2+\log Y}{2\sqrt{Y}}\|f\|_{W_3}$. Similarly, $\|S_Y d\pi(H) f\|_{W_3} \ll \frac{\log Y}{\sqrt{Y}}\|f\|_{W_4}$ and $\|T_y d\pi\,(X_-)\, f\|_{W_3} \ll \frac{\log y}{\sqrt{y}}\|f\|_{W_4}$ for all $y \geq 1$. Combining with the integral formula 11 and assuming $\mathcal{Y} = T$, we have explicit bound of the Sobolev norm,

$$
\begin{aligned}
& \int_{B_\infty} ||F_{\mathcal{Y}}(\pi_{\infty,\xi}(h_\infty a_\infty(\mathcal{Y}))\phi^{(n)}_{\infty,\xi})||_{W_3} \\
& -\frac{1}{2}||S_{\mathcal{Y}} d\pi_{\infty,\xi}(Y)(\pi_{\infty,\xi}(h_\infty a_\infty(\mathcal{Y}))\phi^{(n)}_{\infty,\xi})||_{W_3}\cdot ht(h_\infty a_\infty(\mathcal{Y})\Gamma(g_\infty, g_p))^{1/2}\, d\nu_\infty(h_\infty) \\
& \leq (1+n^2+|s(\xi)|^2)^2(1+\log T)\sqrt{T}\cdot ht((a_\infty(T), Id_p)\Gamma(g_\infty, g_p))^{1/2}
\end{aligned}
$$

Here we need a lemma on the property of height function since the point changes within the integral formula.

and the last part of the integral formula has bound,

$$
\begin{aligned}
& \int_0^{\mathcal{Y}} ||T_{\mathcal{Y}} d\pi_{\infty,\xi}(X_-)(\pi_{\infty,\xi}(h_\infty a_\infty(\mathcal{Y}))\phi^{(n)}_{\infty,\xi})||_{W_3}\nu_\infty(h_\infty) \\
& \leq (1+n^2+|s(\xi)|^2)^2 \int_0^T \frac{\log y}{\sqrt{y}}\cdot ht(a_\infty(T)\Gamma(g_\infty, g_p))^{1/2}\, dy
\end{aligned}
$$

As for the p-adic integral $\int_{B_p} ||\pi_{p,\xi} f_{p,\xi}||_2\, dh_p$ is bounded by

$$\int_{B_p} ||\pi_{p,\xi} f_{p,\xi}||_2\, dh_p \ll d_W \text{the bound from matrix coefficient has to be rewritten.}$$

Finally we obtain the bound of the average when $\xi \in Z^{ct}$,

$$\int_B ||\pi((h_\infty, h_p))f||_{W_3}\, d\nu_\infty(h_\infty)\nu_p(h_p)$$
$$\ll_\Gamma \nu_\infty(B_\infty)\nu_p(B_p)^{-1/2} d_W \left(\frac{T}{ht(a_\infty(T)\Gamma(g_\infty, g_p))}\right)^{-\frac{1}{2}} (\log T)^2 \Xi(a_n)^{-s_1}||f||_{W_4}$$

The next step is considering $\xi \in Z^0_+ \cup Z^{rs}$, in this case it becomes simpler since $Z^0_+ \cup Z^{rs}$ are finite, which means that every $\mathcal{H}_\xi$ is a closed subspace. Similarly, we want to estimate the component of the integral formula, where the detailed argument about the spectral gap can be found in [16]. And also $\|F_Y f\|_{W_3} \ll_\Gamma Y^{s-1}\|f\|_{W_3}$, $\|S_Y d\pi(H) f\|_{W_3} \ll_\Gamma Y^{s-1}\|f\|_{W_4}$, and $\|T_y d\pi\,(X_-)\, f\|_{W_3} \ll_\Gamma y^{s-1}\|f\|_{W_4}$ for all $y \geqq 1$.
Then the horocycle average with function $f \in \oplus_{\xi \in Z^0_+ \cup Z^{rs}} \mathcal{H}_\xi$ is bounded,

$$\left|\frac{1}{\nu(B)} \int_B f((h_\infty, h_p)\Gamma(g_\infty, g_p))\, d\nu_\infty(h_\infty)\nu_p(h_p)\right|$$
$$\ll \nu_\infty(B)^{s_1-1}\nu_p(B_p)^{-1/2} d_W \Xi(a_n)^{1-s_1}||f||_{W_4}$$

At the end, we study the case $f \in \mathcal{H}^-_o$, by a similar argument as in Burger[1], It follows from Lemma 2.2 that the supremum norm $N(f) = \sup_{p \in \Gamma\backslash G} |f(p)|$ is a well-defined and continuous function on $\mathcal{H}^-_o \cap \mathcal{H}^\infty$. Write

$$v_{\varepsilon,\xi} = \frac{1}{T}\int_\varepsilon^Y [1 - \pi(\mathbf{n}(T))]\pi(\mathbf{a}(y)) T_y \left(d\pi_{\infty,\xi}\,(X_-)\,\phi^{(n)}_{\infty,\xi}\right) dy \in \mathcal{H}^-_o \cap \mathcal{H}^\infty$$

then the last line in (23) is the same as $v_0 = \lim_{\varepsilon \to 0+} v_\varepsilon$ (limit in the norm $\|\cdot\|$). The norm N is clearly invariant under $\pi(g)$, for all $g \in G$, and hence by Lemma 2.2 and (22)

$$N\left(\nu_{\varepsilon,\xi}\right) \leqq \frac{2}{T}\int_\varepsilon^Y N\left(T_y d\pi_{\infty,\xi}\,(X_-)\, f\right) dy$$
$$\ll_\Gamma \frac{2}{T}\int_\varepsilon^Y \|T_y d\pi_{\infty,\xi}\,(X_-)\, f\|_{W_3}\, dy \ll \frac{\sqrt{Y}\log(Y+1)}{T} \cdot \|f\|_{W_4}.$$

Similar estimates also show that $\{v_{j^{-1}}\}_{j=1}^\infty$ is a Cauchy sequence with respect to the norm $N(\cdot)$. Hence there is a function w in $C_b(\Gamma\backslash G)$, the space of bounded continuous functions on X, such that $N\left(v_{j^{-1}} - w\right) \to 0$ as $j \to \infty$. But using the fact that $\|v\| \leqq \sqrt{\mathrm{vol}(X)} \cdot N(\nu)$ for all $v \in C_b(\Gamma\backslash G)$, we see that we must have $w = v_0$, and hence $N\left(v_0\right) = \lim_{j\to\infty} N\left(v_{j^{-1}}\right) \ll T^{-1}\sqrt{Y}\log(Y+1) \cdot \|f\|_{W_4}$.
In the meanwhile, the other component in the integral formula is dealt more easily since they are convergent in $\mathcal{H}^\infty$.

$$\left|\frac{1}{\nu(B)}\int_B f((h_\infty,h_p)\Gamma(g_\infty,g_p))\,d\nu_\infty(h_\infty)\nu_p(h_p)\right|$$
$$\ll \left(1+\frac{\log(T(T+1))}{\sqrt{T}}\right))\nu_p(B_p)^{-1/2}d_W\Xi(a_n)^{1-s_1}||f||_{W_4}$$

$\square$

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